Simply Typed Lambda-Calculus

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Deductive Reasoning

The essence

Knowledge obtained via assumptions and logical rules

Studied since Aristotle …

… long before the age of artificial computers

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So what does it have to do with programming ?

A Basic Deductive System

$\mathbb{A}, \mathbb{B} \ldots$ denote propositions and 1 a proposition that always holds

If $\mathbb{A}$ and $\mathbb{B}$ are propositions then

- $\mathbb{A} \times \mathbb{B}$ is a proposition – conjunction of $\mathbb{A}$ and $\mathbb{B}$
- $\mathbb{A} \to \mathbb{B}$ is a proposition – implication of $\mathbb{B}$ from $\mathbb{A}$

A Basic Deductive System

Γ denotes a list of propositions (often called context)

$\Gamma \vdash \mathbb{A}$ reads *"if the propositions in Γ hold then $\mathbb{A}$ also holds"*

$$\frac{\mathbb{A} \in \Gamma}{\Gamma \vdash \mathbb{A}}\ (\text{ass}) \qquad \frac{}{\Gamma \vdash 1}\ (\text{trv}) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{A}}\ (\pi_1) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{B}}\ (\pi_2)$$

$$\frac{\Gamma \vdash \mathbb{A} \qquad \Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \times \mathbb{B}}\ (\text{prd}) \qquad \frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \to \mathbb{B}}\ (\text{cry}) \qquad \frac{\Gamma \vdash \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}}\ (\text{app})$$

Exercise

Show that $\mathbb{A} \times \mathbb{B} \vdash \mathbb{B} \times \mathbb{A}$

New Knowledge From Old

The rules below are derivable from the previous system

$$\frac{\Gamma, \mathbb{A}, \mathbb{B}, \Delta \vdash \mathbb{C}}{\Gamma, \mathbb{B}, \mathbb{A}, \Delta \vdash \mathbb{C}} \text{ (exchange)} \qquad \frac{\Gamma \vdash \mathbb{A}}{\Gamma, \mathbb{B} \vdash \mathbb{A}} \text{ (weakening)}$$

$$\frac{\Gamma, \mathbb{A} \vdash \mathbb{B} \qquad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (cut elimination)}$$

Exercise

Derive the following judgements

- $\mathbb{A} \to \mathbb{B}, \mathbb{B} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{C}$
- $\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$

Back to programming …

The Bare Essentials of Programming

The basic features of programming

- variables
- function application and creation
- pairing ...

Simply-typed λ-calculus

The Bare Essentials of Programming

The basic features of programming

- variables
- function application and creation
- pairing …

Simply-typed λ-calculus

The basis of `Haskell`, `ML`, `Eff`, `F#`, `Agda`, `Elm` and many other programming languages

Simply-typed Lambda-Calculus

Types defined by $\mathbb{A} ::= 1 \mid \mathbb{A} \times \mathbb{A} \mid \mathbb{A} \to \mathbb{A}$

Γ now a non-repetitive list of typed variables $x_1 : \mathbb{A}_1 \ldots x_n : \mathbb{A}_n$

Programs built according to the following deduction rules

$$\frac{x : \mathbb{A} \in \Gamma}{\Gamma \vdash x : \mathbb{A}}\ (\mathrm{ass}) \qquad \frac{}{\Gamma \vdash * : 1}\ (\mathrm{triv}) \qquad \frac{\Gamma \vdash t : \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \pi_1\, t : \mathbb{A}}\ (\pi_1)$$

$$\frac{\Gamma \vdash t : \mathbb{A} \qquad \Gamma \vdash s : \mathbb{B}}{\Gamma \vdash \langle t, s \rangle : \mathbb{A} \times \mathbb{B}}\ (\mathrm{prd}) \qquad \frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}. t : \mathbb{A} \to \mathbb{B}}\ (\mathrm{cry})$$

$$\frac{\Gamma \vdash t : \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash t\, s : \mathbb{B}}\ (\mathrm{app})$$

Examples of λ-terms

$x : \mathbb{A} \vdash x : \mathbb{A}$ (identity)

$x : \mathbb{A} \vdash \langle x, x \rangle : \mathbb{A} \times \mathbb{A}$ (duplication)

$x : \mathbb{A} \times \mathbb{B} \vdash \langle \pi_2\, x, \pi_1\, x \rangle : \mathbb{B} \times \mathbb{A}$ (swap)

$f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{C} \vdash \lambda x : \mathbb{A}.\, g(f\, x) : \mathbb{A} \to \mathbb{C}$ (composition)

Exercises

Recall the derivations that lead to the judgement

$$\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$$

Build the corresponding program

Derive as well the judgement

$$\mathbb{A} \to \mathbb{B} \vdash \mathbb{A} \times \mathbb{C} \to \mathbb{B} \times \mathbb{C}$$

and subsequently build the corresponding program

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We wish to assign a mathematical meaning to λ-terms

$$[\![-]\!] : \lambda\text{-terms} \longrightarrow \ldots$$

so that we can reason about them rigorously, and take advantage of known mathematical theories

We wish to assign a mathematical meaning to λ-terms

$$[\![-]\!] : \lambda\text{-terms} \longrightarrow \dots$$

so that we can reason about them rigorously, and take advantage of known mathematical theories

This is the goal of the next slides. But first ...

Functions: Basic Facts

For every set X there exists a 'trivial' function

$$! : X \longrightarrow \{\star\} = 1 \qquad !\,(x) = \star$$

We can always pair two functions into $f : X \to A$, $g : X \to B$

$$\langle f, g \rangle : X \to A \times B \qquad \langle f, g \rangle(x) = (f\,x, g\,x)$$

There exist projection functions

$$\pi_1 : X \times Y \to X \qquad \pi_1(x, y) = x$$
$$\pi_2 : X \times Y \to Y \qquad \pi_2(x, y) = y$$

Functions: Basic Facts

We can always curry a function $f : X \times Y \to Z$ into

$$\lambda f : X \to Z^Y \qquad \lambda f(x) = (y \mapsto f(x, y))$$

Consider sets X, Y, Z. There exists an application function

$$\mathrm{app} : Z^Y \times Y \to Z \qquad \mathrm{app}(f, y) = f\, y$$

Denotational Semantics

Types $\mathbb{A}$ interpreted as sets $[\![\mathbb{A}]\!]$

$$[\![1]\!] = \{\star\}$$
$$[\![\mathbb{A} \times \mathbb{B}]\!] = [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]$$
$$[\![\mathbb{A} \to \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

Typing contexts Γ interpreted as Cartesian products

$$[\![\Gamma]\!] \; = [\![x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n]\!] \; = [\![\mathbb{A}_1]\!] \times \cdots \times [\![\mathbb{A}_n]\!]$$

Terms $\Gamma \vdash t : \mathbb{A}$ interpreted as functions

$$[\![\Gamma \vdash t : \mathbb{A}]\!] \; : [\![\Gamma]\!] \; \longrightarrow [\![\mathbb{A}]\!]$$

Term $\Gamma \vdash t : \mathbb{A}$ interpreted as a function

$$[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i : \mathbb{A}]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = !} \qquad \frac{[\![\Gamma \vdash t : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 t : \mathbb{A}]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash s : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle t, s\rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g\rangle} \qquad \frac{[\![\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}. t : \mathbb{A} \to \mathbb{B}]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A} \to \mathbb{B}]\!] = f \quad [\![\Gamma \vdash s : \mathbb{A}]\!] = g}{[\![\Gamma \vdash t\, s : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g\rangle}$$

The Unravelling

$$\llbracket x \vdash \langle \pi_2\, x,\ \pi_1\, x \rangle \rrbracket = \ldots$$

$$\llbracket - \vdash \lambda x.\, \langle \pi_2\, x,\ \pi_1\, x \rangle \rrbracket = \ldots$$

$$\llbracket f, g, x \vdash g\, f\, x \rrbracket = \ldots$$

$$\llbracket f, g \vdash \lambda x.\, g\, f\, x \rrbracket = \ldots$$

$$\llbracket f, x \vdash \langle f\, \pi_1\, x,\ \pi_2\, x \rangle \rrbracket = \ldots$$

$$\llbracket f \vdash \lambda x.\, \langle f\, \pi_1\, x,\ \pi_2\, x \rangle \rrbracket = \ldots$$

$$\llbracket - \vdash \lambda f.\, \lambda x.\, \langle f\, \pi_1\, x,\ \pi_2\, x \rangle \rrbracket = \ldots$$

(N.B. all types omitted for simplicity)

Show that the following equations hold

$$\llbracket x, y \vdash \pi_1 \langle x, y \rangle \rrbracket = \llbracket x, y \vdash x \rrbracket$$

$$\llbracket \Gamma \vdash t \rrbracket = \llbracket \Gamma \vdash \langle \pi_1\ t, \pi_2\ t \rangle \rrbracket$$

$$\llbracket x \vdash (\lambda y.\, \langle x, y \rangle)\ x \rrbracket = \llbracket x \vdash \langle x, x \rangle \rrbracket$$