Simply-typed λ-calculus: a brief overview

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The Four Chapters of this course

Design and analysis of communicating systems

Extension to the real-time setting

Going full cyber-physical

Programming with algebraic effects . . .

a uniform approach to the previous chapters

Overview

Modern programming typically involves different effects

- (internal) state manipulation
- communication
- probabilistic operations
- cyber-physical behaviour

We will study the mathematical foundations of

Programming with effects

Deductive reasoning

Deductive reasoning

Assumptions + rules = new knowledge

Example

If P implies Q and Q implies R then P implies R

Deductive reasoning has been studied since Aristotle . . . long before the age of artificial computers

So what does it have to do with programming?

A basic deductive system pt. I

Let $\mathbb{A}, \mathbb{B} \ldots$ denote propositions (a property or a statement) and 1 denote a proposition that always holds. If $\mathbb{A}$ and $\mathbb{B}$ are propositions then

- $\mathbb{A} \times \mathbb{B}$ is a proposition – it denotes the conjunction of $\mathbb{A}$ and $\mathbb{B}$
- $\mathbb{A} \to \mathbb{B}$ is a proposition – it tells that $\mathbb{A}$ implies $\mathbb{B}$

A basic deductive system pt. II

Γ denotes a list of propositions. $\Gamma \vdash \mathbb{A}$ means "if the propositions in Γ hold then (we deduce that) $\mathbb{A}$ also holds"

$$\frac{\mathbb{A} \in \Gamma}{\Gamma \vdash \mathbb{A}}\ (\text{ass}) \qquad \frac{}{\Gamma \vdash 1}\ (\text{trv}) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{A}}\ (\pi_1) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{B}}\ (\pi_2)$$

$$\frac{\Gamma \vdash \mathbb{A} \qquad \Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \times \mathbb{B}}\ (\text{prd}) \qquad \frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \to \mathbb{B}}\ (\text{cry}) \qquad \frac{\Gamma \vdash \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}}\ (\text{app})$$

Exercise

Show that $\mathbb{A} \times \mathbb{B} \vdash \mathbb{B} \times \mathbb{A}$

Building new rules from the original ones

The following rules are derivable from the previous system

$$\frac{\Gamma \vdash \mathbb{A}}{\Gamma, \mathbb{B} \vdash \mathbb{A}} \qquad \frac{\Gamma, \mathbb{A}, \mathbb{B}, \Delta \vdash \mathbb{C}}{\Gamma, \mathbb{B}, \mathbb{A}, \Delta \vdash \mathbb{C}}$$

Exercise

Prove that $\mathbb{A} \to \mathbb{B}, \mathbb{B} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{C}$ and also that $\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$. Are these deductions familiar?

Going back to programming ...

The essentials of programming

In order to study effectful programming, we should think of what are the basic features of (higher-order) programming ...

- variables
- function application and abstraction
- pairing ...

and base our study on the simplest programming language containing these features ...

Simply-typed λ-calculus

The basis of `Haskell`, `ML`, `Eff`, `F#`, `Agda`, `Elm` and many other programming languages

Simply-typed λ-Calculus

Types $\mathbb{A} \ni 1 \mid \mathbb{A} \times \mathbb{A} \mid \mathbb{A} \to \mathbb{A}$

Γ is now a non-repetitive list of typed variables $x_1 : \mathbb{A}_1 \ldots x_n : \mathbb{A}_n$

Programs are built according to the previous deduction rules

$$\frac{x : \mathbb{A} \in \Gamma}{\Gamma \vdash x : \mathbb{A}}\ (\mathrm{ass}) \qquad \frac{}{\Gamma \vdash * : 1}\ (\mathrm{triv}) \qquad \frac{\Gamma \vdash V : \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \pi_1 V : \mathbb{A}}\ (\pi_1)$$

$$\frac{\Gamma \vdash V : \mathbb{A} \qquad \Gamma \vdash U : \mathbb{B}}{\Gamma \vdash \langle V, U \rangle : \mathbb{A} \times \mathbb{B}}\ (\mathrm{prd}) \qquad \frac{\Gamma, x : \mathbb{A} \vdash V : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}.\, V : \mathbb{A} \to \mathbb{B}}\ (\mathrm{cry})$$

$$\frac{\Gamma \vdash V : \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash U : \mathbb{A}}{\Gamma \vdash V\, U : \mathbb{B}}\ (\mathrm{app})$$

Examples of λ-terms

$x : \mathbb{A} \vdash x : \mathbb{A}$ (identity)

$x : \mathbb{A} \vdash \langle x, x \rangle : \mathbb{A} \times \mathbb{A}$ (duplication)

$x : \mathbb{A} \times \mathbb{B} \vdash \langle \pi_2\ x, \pi_1\ x \rangle : \mathbb{B} \times \mathbb{A}$ (swap)

$f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{C} \vdash \lambda x : \mathbb{A}.\ g(f\ x) : \mathbb{A} \to \mathbb{C}$ (composition)

Exercise

Build a λ-term $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{A} \to \mathbb{C} \vdash ? : \mathbb{A} \to \mathbb{B} \times \mathbb{C}$ that pairs the outputs given by f and g

Semantics for simply-typed λ-calculus

We wish to assign a mathematical meaning to λ-terms

$$[\![-]\!] \colon \lambda\text{-terms} \longrightarrow \ldots$$

so that we can reason about them in a rigorous way, and take advantage of known mathematical theories

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This is the goal of the next slides: we will study how to interpret λ-terms as functions. But first . . .

Basic facts about functions

For every set X, there is a ‘trivial’ function

$$! : X \longrightarrow \{\star\} = 1, \qquad !\,(x) = \star$$

We can always pair two functions $f : X \to A$, $g : X \to B$ into

$$\langle f, g \rangle : X \to A \times B, \qquad \langle f, g \rangle(x) = (f\,x, g\,x)$$

Consider two sets X, Y. There exist ‘projection’ functions

$$\pi_1 : X \times Y \to X, \qquad \pi_1(x, y) = x$$
$$\pi_2 : X \times Y \to Y, \qquad \pi_2(x, y) = y$$

Basic facts about functions

We can always 'curry' a function $f : X \times Y \to Z$ into

$$\lambda f : X \to Z^Y, \qquad \lambda f(x) = (y \mapsto f(x, y))$$

Consider sets X, Y, Z. There exists an 'application' function

$$\mathrm{app} : Z^Y \times Y \to Z, \qquad \mathrm{app}(f, y) = f\ y$$

Functional semantics for the simply-typed λ-calculus

Types $\mathbb{A}$ are interpreted as sets $[\![\mathbb{A}]\!]$

$$[\![1]\!] = \{\star\}$$
$$[\![\mathbb{A} \times \mathbb{B}]\!] = [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]$$
$$[\![\mathbb{A} \to \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

A typing context Γ is interpreted as

$$[\![\Gamma]\!] = [\![x_1 : \mathbb{A}_1 \times \cdots \times x_n : \mathbb{A}_n]\!] = [\![\mathbb{A}_1]\!] \times \cdots \times [\![\mathbb{A}_n]\!]$$

A λ-term $\Gamma \vdash V : \mathbb{A}$ is interpreted as a function

$$[\![\Gamma \vdash V : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

Functional semantics for the simply-typed λ-calculus

A λ-term $\Gamma \vdash V : \mathbb{A}$ is interpreted as a function

$$[\![\Gamma \vdash V : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

in the following way

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i : \mathbb{A}]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = !} \qquad \frac{[\![\Gamma \vdash V : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 V : \mathbb{A}]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash V : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash U : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle V, U \rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g \rangle} \qquad \frac{[\![\Gamma, x : \mathbb{A} \vdash V : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}.\, V : \mathbb{A} \to \mathbb{B}]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash V : \mathbb{A} \to \mathbb{B}]\!] = f \quad [\![\Gamma \vdash U : \mathbb{A}]\!] = g}{[\![\Gamma \vdash V\, U : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g \rangle}$$

Exercises

Show that the following equations hold.

$$
\begin{aligned}
[\![x : \mathbb{A}, y : \mathbb{B} \vdash \pi_1 \langle x, y \rangle : \mathbb{A}]\!] &= [\![x : \mathbb{A}, y : \mathbb{B} \vdash x : \mathbb{A}]\!] \\
[\![\Gamma \vdash V : \mathbb{A}]\!] &= [\![\Gamma \vdash \langle \pi_1 V, \pi_2 V \rangle : \mathbb{A}]\!] \\
[\![x : \mathbb{A} \vdash (\lambda y.\, \langle x, y \rangle)\, x : \mathbb{A} \times \mathbb{A}]\!] &= [\![x : \mathbb{A} \vdash \langle x, x \rangle : \mathbb{A} \times \mathbb{A}]\!]
\end{aligned}
$$