MFP/CSI/2627 — Relation Calculus Reference Sheet

Relational composition

Pointwise def.

$$B \xleftarrow{R} A \xleftarrow{S} C \qquad b(R \cdot S)c \;\equiv\; \langle \exists\, a \;:\; b\, R\, a :\; a\, S\, c \rangle \tag{1}$$

Associativity

$$R \cdot (S \cdot P) = (R \cdot S) \cdot P \tag{2}$$

Identity

$$\begin{cases} R = R \cdot id_A \\ R = id_B \cdot R \end{cases} \tag{3}$$

Converse

Pointwise def.

$$b\, R\, a \quad \Leftrightarrow \quad a\, R^{\circ}\, b \tag{4}$$

Universal-$^{\circ}$

$$X^{\circ} \subseteq Y \quad \equiv \quad X \subseteq Y^{\circ} \tag{5}$$

Involution

$$(R^{\circ})^{\circ} = R \tag{6}$$

Contravariance

$$(R \cdot S)^{\circ} = S^{\circ} \cdot R^{\circ} \tag{7}$$

Isomorphism

$$R \subseteq S \equiv R^{\circ} \subseteq S^{\circ} \tag{8}$$

"Guardanapo"

$$b(f^{\circ} \cdot R \cdot g)a \quad \equiv \quad (f\, b)R(g\, a) \tag{9}$$

Relation inclusion

Pointwise

$$R \subseteq S \quad \textit{iff} \quad \langle \forall\, a, b \;::\; b\, R\, a \;\Rightarrow\; b\, S\, a \rangle \tag{10}$$

Reflexion

$$R \subseteq R \tag{11}$$

Transitivity

$$R \subseteq S \wedge S \subseteq Q \quad \Rightarrow \quad R \subseteq Q \tag{12}$$

Top and bottom

$$\bot \;\subseteq\; R \;\subseteq\; \top \tag{13}$$

Absorption

$$R \cdot \bot = \bot \cdot R = \bot \tag{14}$$

RELATION EQUALITY

Pointwise

$$R = S \quad \textit{iff} \quad \langle \forall\, a, b \,:\, a \in A \wedge b \in B :\; b\, R\, a \;\Leftrightarrow\; b\, S\, a \rangle \tag{15}$$

"Ping-pong"

$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R \tag{16}$$

Indirect equality

$$R = S \quad \equiv \quad \langle \forall\, X \,::\, (X \subseteq R \;\Leftrightarrow\; X \subseteq S) \rangle \tag{17}$$

$$\equiv \quad \langle \forall\, X \,::\, (R \subseteq X \;\Leftrightarrow\; S \subseteq X) \rangle \tag{18}$$

Typical layouts of indirect equality proofs

$$\left\{ \begin{array}{ll} & X \subseteq R \\ \equiv & \quad \{\ \ldots\ \} \\ & X \subseteq \ldots \\ \equiv & \quad \{\ \ldots\ \} \\ & X \subseteq S \\ :: & \quad \{\ \text{indirect equality}\ \} \\ & R = S \\ \Box & \end{array} \right. \qquad \left\{ \begin{array}{ll} & R \subseteq X \\ \equiv & \quad \{\ \ldots\ \} \\ & \ldots \subseteq X \\ \equiv & \quad \{\ \ldots\ \} \\ & S \subseteq X \\ :: & \quad \{\ \text{indirect equality}\ \} \\ & R = S \\ \Box & \end{array} \right.$$

RELATION TAXONOMY

Kernel

$$\mathsf{ker}\ R = R^{\circ} \cdot R \tag{19}$$

Image

$$\mathsf{img}\ R \stackrel{\text{def}}{=} R \cdot R^{\circ} \tag{20}$$

Duality

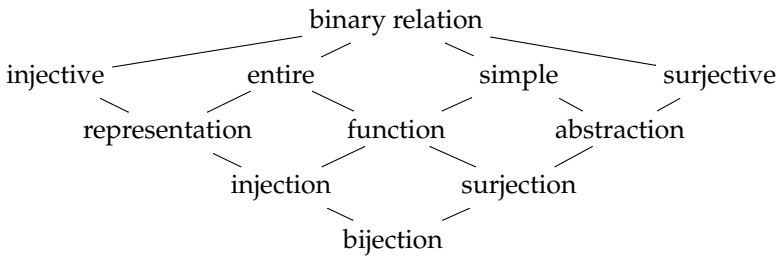
$$\mathsf{ker}\ (R^{\circ}) \;=\; \mathsf{img}\ R \tag{21}$$

$$\mathsf{img}\ (R^{\circ}) \;=\; \mathsf{ker}\ R \tag{22}$$

Criteria

	Reflexive	*Coreflexive*
ker R	entire R	injective R
img R	surjective R	simple R

(23)



Taxonomy using matrices

- **entire** — at least one 1 in every column
- **surjective** — at least one 1 in every row
- **simple** — at most one 1 in every column
- **injective** — at most one 1 in every row
- **bijective** — exactly one 1 in evey column and every row.

Difunctional R

$$R \cdot R^\circ \cdot R \subseteq R \tag{24}$$

FUNCTIONS

Shunting rules

$$f \cdot R \subseteq S \equiv R \subseteq f^\circ \cdot S \tag{25}$$
$$R \cdot f^\circ \subseteq S \equiv R \subseteq S \cdot f \tag{26}$$

Equality

$$f \subseteq g \equiv f = g \equiv f \supseteq g \tag{27}$$

CONSTANT FUNCTIONS

Natural property

$$\underline{k} \cdot R \subseteq \underline{k} \tag{28}$$

Corollary

$$\ker\ ! = \frac{!}{!} = \top \tag{29}$$

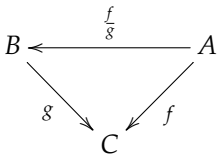
Truth functions

$$true = \underline{\text{TRUE}} \tag{30}$$
$$false = \underline{\text{FALSE}} \tag{31}$$

FUNCTION DIVISION

Definition

$$\frac{f}{g} = g^\circ \cdot f \quad cf. \tag{32}$$



Properties

$$\frac{f}{id} = f \tag{33}$$
$$\left(\frac{f}{g}\right)^\circ = \frac{g}{f} \tag{34}$$

$$\frac{f \cdot h}{g \cdot k} \quad = \quad k^\circ \cdot \frac{f}{g} \cdot h \tag{35}$$

$$\frac{f}{f} \quad = \quad \ker f \tag{36}$$

$$a \neq b \quad \Leftrightarrow \quad \frac{a}{b} = \bot \tag{37}$$

Relation union

Pointwise definition

$$b\ (R \cup S)\ a \quad \equiv \quad b\ R\ a \vee b\ S\ a \tag{38}$$

Universal property

$$R \cup S \ \subseteq \ X \quad \equiv \quad R \subseteq X \ \wedge \ S \subseteq X \tag{39}$$

Right linearity

$$R \cdot (S \cup Q) \quad = \quad (R \cdot S) \cup (R \cdot Q) \tag{40}$$

Left linearity

$$(S \cup Q) \cdot R \quad = \quad (S \cdot R) \cup (T \cdot R) \tag{41}$$

Converse-$\cup$

$$(R \cup S)^\circ = R^\circ \cup S^\circ \tag{42}$$

Top-$\cup$

$$R \cup \top = \top \tag{43}$$

Bottom-$\cup$

$$R \cup \bot = R \tag{44}$$

Union simplicity

$$M \cup N \text{ is simple} \quad \equiv \quad M, N \text{ are simple and } M \cdot N^\circ \subseteq id \tag{45}$$

Relation intersection

Pointwise definition

$$b\ (R \cap S)\ a \quad \equiv \quad b\ R\ a \wedge b\ S\ a \tag{46}$$

Universal property

$$X \ \subseteq \ R \cap S \quad \equiv \quad X \subseteq R \ \wedge \ X \subseteq S \tag{47}$$

Distribution (1)

$$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \quad \Leftarrow \quad \begin{pmatrix} Q \cdot \mathsf{img}\ R \subseteq Q \\ \vee \\ S \cdot \mathsf{img}\ R \subseteq S \end{pmatrix} \tag{48}$$

Distribution (2)

$$R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \quad \Leftarrow \quad \begin{pmatrix} (\ker R) \cdot Q \subseteq Q \\ \vee \\ (\ker R) \cdot S \subseteq S \end{pmatrix} \tag{49}$$

Distribution (3)

$$g^\circ \cdot (R \cap S) \cdot f \quad = \quad g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f \tag{50}$$

Converse-$\cap$

$$(R \cap S)^\circ = R^\circ \cap S^\circ \tag{51}$$

Bottom-$\cap$

$$R \cap \bot = \bot \tag{52}$$

Top-$\cap$

$$R \cap \top = R \tag{53}$$

Misc.

$$k^\circ \cdot (f \cup g) = \frac{f}{k} \cup \frac{g}{k} \quad , \quad k^\circ \cdot (f \cap g) = \frac{f}{k} \cap \frac{g}{k} \tag{54}$$

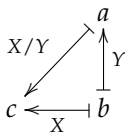
Relation division

Universal-/

$$Z \cdot Y \subseteq X \;\equiv\; Z \subseteq X/Y \tag{55}$$

Pointwise-/

$$c\ (X/Y)\ a \;\equiv\; \langle \forall\ b\ :\ a\ Y\ b :\ c\ X\ b \rangle \tag{56}$$



Universal-$\backslash$

$$X \cdot Z \subseteq Y \;\Leftrightarrow\; Z \subseteq X \setminus Y \tag{57}$$

Pointwise-$\backslash$

$$a(X \setminus Y)c \quad \equiv \quad \langle \forall\ b\ :\ b\ X\ a :\ b\ Y\ c \rangle \tag{58}$$

Duality

$$(X \setminus Y)^\circ \quad = \quad Y^\circ / X^\circ \tag{59}$$

Misc.

$$X \cdot f \quad = \quad X/f^\circ \tag{60}$$

$$f \setminus X \quad = \quad f^\circ \cdot X \tag{61}$$

$$X/\bot \quad = \quad \top \tag{62}$$

$$X/id \quad = \quad X \tag{63}$$

$$R \setminus (f^\circ \cdot S) \quad = \quad f \cdot R \setminus S \tag{64}$$

$$R \setminus \top \cdot S \quad = \quad !\cdot R \setminus !\cdot S \tag{65}$$

$$R \,/\, (S \cup P) \quad = \quad (R \,/\, S) \cap (R \,/\, P) \tag{66}$$

RELATION DIFFERENCE, IMPLICATION AND NEGATION

Universal-$(-)$

$$X - R \subseteq Y \quad \equiv \quad X \subseteq Y \cup R \tag{67}$$

Pointwise-$\Rightarrow$

$$b(R \Rightarrow S)a \quad \equiv \quad (b\ R\ a) \Rightarrow (b\ S\ a) \tag{68}$$

Universal-$\Rightarrow$

$$R \cap X \subseteq Y \quad \equiv \quad X \subseteq (R \Rightarrow Y) \tag{69}$$

Distribution

$$f^\circ \cdot (R \Rightarrow S) \cdot g \quad = \quad (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g) \tag{70}$$

Definition-$\neg$

$$\neg R = (R \Rightarrow \bot) \tag{71}$$

Pointwise-$\neg$

$$b\ (\neg R)\ a \ \Leftrightarrow \ \neg\ (b\ R\ a) \tag{72}$$

Complementation

$$R \cup \neg R = \top \tag{73}$$

Difference versus implication

$$\top - R \ \subseteq \ R \Rightarrow \bot \tag{74}$$

de Morgan

$$\neg(R \cup S) = (\neg R) \cap (\neg S) \tag{75}$$

Schröder's rule

$$\neg Q \cdot S^\circ \ \subseteq \ \neg R \ \Leftrightarrow \ R^\circ \cdot \neg Q \ \subseteq \ \neg S \tag{76}$$

RELATION SHRINKING

Definition-shrink

$$R \restriction S \ = \ R \cap S/R^\circ \tag{77}$$

Universal-shrink

$$X \subseteq R \restriction S \ \equiv \ X \subseteq R \ \wedge \ X \cdot R^\circ \subseteq S \tag{78}$$

MONOTONICITY

Composition

$$R \ \subseteq \ S \wedge Q \ \subseteq \ U \ \Rightarrow \ R \cdot Q \subseteq S \cdot U \tag{79}$$

Converse

$$R \ \subseteq \ S \ \Rightarrow \ R^\circ \ \subseteq \ S^\circ \tag{80}$$

Intersection

$$R \subseteq S \wedge U \subseteq V \ \Rightarrow \ R \cap U \subseteq S \cap V \tag{81}$$

Union

$$R \subseteq S \wedge U \subseteq V \ \Rightarrow \ R \cup U \subseteq S \cup V \tag{82}$$

ENDO-RELATION TAXONOMY

Reflexive

$$R \text{ is reflexive } \equiv \ id \subseteq R \tag{83}$$

Coreflexive

$$R \text{ is coreflexive } \equiv \ R \subseteq id \tag{84}$$

Transitive

$$R \text{ is transitive } \equiv \ R \cdot R \subseteq R \tag{85}$$

Symmetric

$$R \text{ is symmetric } \equiv \ R \subseteq R^{\circ} (\equiv R = R^{\circ}) \tag{86}$$

Anti-symmetric

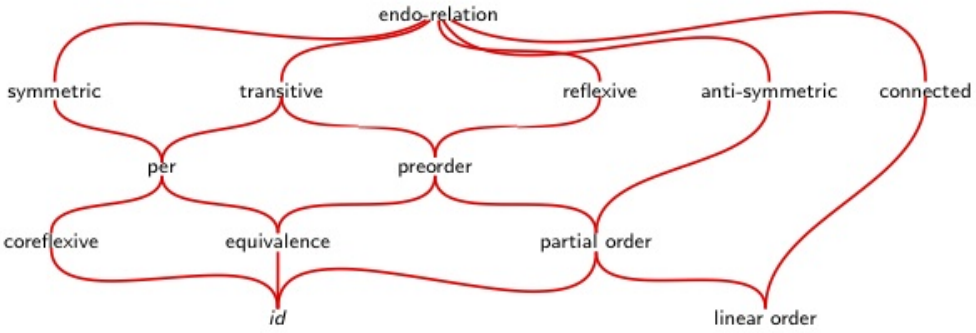
$$R \text{ is anti-symmetric } \equiv \ R \cap R^{\circ} \ \subseteq id \tag{87}$$

Irreflexive

$$R \text{ is irreflexive } \equiv \ R \cap id = \bot \tag{88}$$

Connected

$$R \text{ is connected } \equiv \ R \cup R^{\circ} = \top \tag{89}$$



COREFLEXIVES

Def-coreflexive

$$\Phi_p = id \cap \frac{true}{p} \tag{90}$$

Conv-coreflexive

$$\Phi_p^{\circ} = \Phi_p \tag{91}$$

Cond-coreflexive

$$\Phi_p \cdot \top = \frac{true}{p} \tag{92}$$

Pre-restriction

$$R \cdot \Phi_p \ \ = \ \ R \cap \top \cdot \Phi_p \tag{93}$$

Post-restriction

$$\Phi_q \cdot R \quad = \quad R \cap \Phi_q \cdot \top \tag{94}$$

Coreflexive-meet

$$\Phi_q \cdot \Phi_p = \Phi_q \cap \Phi_p \tag{95}$$

Guards

$$p? = [\Phi_p, \Phi_{\neg\, p}]^\circ \tag{96}$$

McCarthy

$$p \to R, S = [R, S] \cdot p? \tag{97}$$

Domain

$$\delta\, R = \mathsf{ker}\, R \cap id = \top \cdot R \cap id \tag{98}$$

Range

$$\rho\, R = \delta\, (R^\circ) \tag{99}$$

Domain-universal

$$\delta\, X \;\subseteq\; \Phi \equiv X \subseteq \top \cdot \Phi \quad (\Phi \text{ coreflexive}) \tag{100}$$

Range-universal

$$\rho\, R \;\subseteq\; \Phi \equiv R \subseteq \Phi \cdot \top \quad (\Phi \text{ coreflexive}) \tag{101}$$

Dom-comp

$$\delta\, (R \cdot S) = \delta\, (\delta\, R \cdot S) \tag{102}$$

Comp-dom

$$R = R \cdot \delta\, R \tag{103}$$

PAIRING ("SPLITS")

Pairing-pointwise

$$(a, b)\ \langle R, S\rangle\ c \;\Leftrightarrow\; a\ R\ c \wedge b\ S\ c \tag{104}$$

Pairing-def

$$\langle R, S\rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S \tag{105}$$

Universal-pairing

$$X \subseteq \langle R, S\rangle \quad \Leftrightarrow \quad \begin{cases} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{cases} \tag{106}$$

Pairing-fusion

$$\begin{array}{l} \langle R, S\rangle \cdot Q = \langle R \cdot Q, S \cdot Q\rangle \\ \qquad \Leftarrow\ R \cdot (\mathsf{img}\ Q) \subseteq R \vee S \cdot (\mathsf{img}\ Q) \subseteq S \end{array} \tag{107}$$

Pairing-fusion (functions)

$$\langle R, S\rangle \cdot f \;=\; \langle R \cdot f, S \cdot f\rangle \tag{108}$$

Pairing and converse

$$\langle R, S\rangle^\circ \cdot \langle X, Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y) \tag{109}$$

Kernel of pairing

$$\ker \langle R, S \rangle \;=\; \ker R \cap \ker S \tag{110}$$

Functor-$\times$ def

$$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2 \rangle \tag{111}$$

Functor-$\times$-id

$$id \times id = id \tag{112}$$

Functor-$\times$-composition

$$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q) \tag{113}$$

Functor-$\times$ absorption

$$(R \times S) \cdot \langle P, Q \rangle = \langle R \cdot P, S \cdot Q \rangle \tag{114}$$

Functor-$\times$-division

$$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k} \tag{115}$$

Pairing division

$$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f, h \rangle}{\langle g, k \rangle} \tag{116}$$

Tabulation

$$\pi_1 \cdot \pi_2^\circ = \top \tag{117}$$

COPRODUCTS

Universal

$$X = [R, S] \quad \Leftrightarrow \quad \begin{cases} X \cdot i_1 = R \\ X \cdot i_2 = S \end{cases} \tag{118}$$

Definition

$$[R, S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ \tag{119}$$

$+$-Reflexion

$$[i_1, i_2] = id \tag{120}$$

$+$-Fusion

$$R \cdot [S, Q] = [R \cdot S, R \cdot Q] \tag{121}$$

Disjointness

$$i_1^\circ \cdot i_2 = \bot \tag{122}$$

Divide & conquer

$$[R, S] \cdot [Q, U]^\circ \;=\; (R \cdot Q^\circ) \cup (S \cdot U^\circ) \tag{123}$$

Image of either

$$\mathsf{img}\ [R, S] \;=\; \mathsf{img}\ R \cup \mathsf{img}\ S \tag{124}$$

Exchange law

$$[\langle R, S \rangle, \langle Q, V \rangle] \;=\; \langle [R, Q], [S, V] \rangle \tag{125}$$

Functor-+-def

$$R + S = [i_1 \cdot R \, , i_2 \cdot S] \tag{126}$$

+-Absorption

$$[R, S] \cdot (P + Q) = [R \cdot P, S \cdot Q] \tag{127}$$

Functor-+-converse

$$(R + S)^\circ = R^\circ + S^\circ \tag{128}$$

Functor-+-division

$$\frac{f}{g} + \frac{h}{k} = \frac{f + h}{g + k} \tag{129}$$

INJECTIVITY PREORDER

Definition

$$R \leqslant S \quad \equiv \quad \ker S \subseteq \ker R \tag{130}$$

Join

$$\langle R, S \rangle \leqslant X \quad \equiv \quad R \leqslant X \wedge S \leqslant X \tag{131}$$

Cancellation

$$R \leqslant \langle R, S \rangle \quad \text{and} \quad S \leqslant \langle R, S \rangle. \tag{132}$$

Shunting

$$R \cdot g \leqslant S \quad \equiv \quad R \leqslant S \cdot g^\circ \tag{133}$$

Meet

$$X \leqslant [R^\circ, S^\circ]^\circ \quad \Leftrightarrow \quad X \leqslant R \wedge X \leqslant S \tag{134}$$

Either-injective

$$id \leqslant [R, S] \Leftrightarrow \begin{cases} id \leqslant R \\ id \leqslant S \\ R^\circ \cdot S = \bot \end{cases} \tag{135}$$

THUMB RULES

A function f is a bijection iff *its converse f° is a function g* (136)

- converse of *injective* is *simple* (and vice-versa) (137)

- converse of *entire* is *surjective* (and vice-versa) (138)

- *smaller* than injective (simple) is injective (simple) (139)

- *larger* than entire (surjective) is entire (surjective) (140)

$\langle R, id \rangle$ is always *injective*, for whatever R (141)

GALOIS CONNECTIONS

Summary

$(f\ b) \leqslant a \equiv b \sqsubseteq (g\ a)$		
Description	$f = g^\flat$	$g = f^\sharp$
Definition	$f\ b = \bigwedge\{a \mid b \sqsubseteq g\ a\}$	$g\ a = \bigsqcup\{b \mid f\ b \leqslant a\}$
Cancellation	$f(g\ a) \leqslant a$	$b \sqsubseteq g(f\ b)$
Distribution	$f(b \sqcup b') = (f\ b) \vee (f\ b')$	$g(a' \wedge a) = (g\ a') \sqcap (g\ a)$
Monotonicity	$b \sqsubseteq b' \Rightarrow f\ b \leqslant f\ b'$	$a \leqslant a' \Rightarrow g\ a \sqsubseteq g\ a'$

(142)

QUANTIFIER CALCULUS

Trading-$\forall$

$$\langle \forall\, k\ :\ R \wedge S :\ T \rangle \quad = \quad \langle \forall\, k\ :\ R :\ S \Rightarrow T \rangle \tag{143}$$

Trading-$\exists$

$$\langle \exists\, k\ :\ R \wedge S :\ T \rangle \quad = \quad \langle \exists\, k\ :\ R :\ S \wedge T \rangle \tag{144}$$

de Morgan

$$\neg\langle \forall\, k\ :\ R :\ T \rangle \quad = \quad \langle \exists\, k\ :\ R :\ \neg T \rangle \tag{145}$$

de Morgan

$$\neg\langle \exists\, k\ :\ R :\ T \rangle \quad = \quad \langle \forall\, k\ :\ R :\ \neg T \rangle \tag{146}$$

One-point-$\forall$

$$\langle \forall\, k\ :\ k = e :\ T \rangle \quad = \quad T[k := e] \tag{147}$$

One-point-$\exists$

$$\langle \exists\, k\ :\ k = e :\ T \rangle \quad = \quad T[k := e] \tag{148}$$

Nesting-$\forall$

$$\langle \forall\, a, b\ :\ R \wedge S :\ T \rangle \quad = \quad \langle \forall\, a\ :\ R :\ \langle \forall\, b\ :\ S :\ T \rangle \rangle \tag{149}$$

Nesting-$\exists$

$$\langle \exists\, a, b\ :\ R \wedge S :\ T \rangle \quad = \quad \langle \exists\, a\ :\ R :\ \langle \exists\, b\ :\ S :\ T \rangle \rangle \tag{150}$$

Rearranging-$\forall$

$$\langle \forall\, k\ :\ R \vee S :\ T \rangle \quad = \quad \langle \forall\, k\ :\ R :\ T \rangle \wedge \langle \forall\, k\ :\ S :\ T \rangle \tag{151}$$

Rearranging-$\forall$

$$\langle \forall\, k\ :\ R :\ T \wedge S \rangle \quad = \quad \langle \forall\, k\ :\ R :\ T \rangle \wedge \langle \forall\, k\ :\ R :\ S \rangle \tag{152}$$

Rearranging-$\exists$

$$\langle \exists\, k\ :\ R :\ T \vee S \rangle \quad = \quad \langle \exists\, k\ :\ R :\ T \rangle \vee \langle \exists\, k\ :\ R :\ S \rangle \tag{153}$$

Rearranging-$\exists$

$$\langle \exists\, k\ :\ R \vee S :\ T \rangle \quad = \quad \langle \exists\, k\ :\ R :\ T \rangle \vee \langle \exists\, k\ :\ S :\ T \rangle \tag{154}$$

Splitting-$\forall$

$$\langle \forall\, j\ :\ R :\ \langle \forall\, k\ :\ S :\ T \rangle \rangle \quad = \quad \langle \forall\, k\ :\ \langle \exists\, j\ :\ R :\ S \rangle :\ T \rangle \tag{155}$$

Splitting-$\exists$

$$\langle \exists\, j\ :\ R :\ \langle \exists\, k\ :\ S :\ T \rangle \rangle \quad = \quad \langle \exists\, k\ :\ \langle \exists\, j\ :\ R :\ S \rangle :\ T \rangle \tag{156}$$