MFP/CSI/2526 — Relation Calculus Reference Sheet

Relational composition

Pointwise def.

$$\begin{tikzcd} B & A \arrow[l, "R"'] & C \arrow[l, "S"'] \arrow[ll, bend left, "R\cdot S"] \end{tikzcd} \quad b(R\cdot S)c \;\equiv\; \langle \exists\, a \,:\, b\,R\,a : \, a\,S\,c\rangle \tag{1}$$

Associativity

$$R\cdot(S\cdot P) = (R\cdot S)\cdot P \tag{2}$$

Identity

$$\begin{cases} R = R\cdot id_A \\ R = id_B\cdot R \end{cases} \tag{3}$$

Converse

Pointwise def.

$$b\,R\,a \quad\Leftrightarrow\quad a\,R^\circ\,b \tag{4}$$

Universal-$^\circ$

$$X^\circ \subseteq Y \quad\equiv\quad X \subseteq Y^\circ \tag{5}$$

Involution

$$(R^\circ)^\circ = R \tag{6}$$

Contravariance

$$(R\cdot S)^\circ = S^\circ\cdot R^\circ \tag{7}$$

Isomorphism

$$R\subseteq S \equiv R^\circ \subseteq S^\circ \tag{8}$$

"Guardanapo"

$$b(f^\circ\cdot R\cdot g)a \quad\equiv\quad (f\,b)R(g\,a) \tag{9}$$

Relation inclusion

Pointwise

$$R\subseteq S \quad \mathit{iff} \quad \langle \forall\, a,b \,::\, b\,R\,a \,\Rightarrow\, b\,S\,a\rangle \tag{10}$$

Reflexion

$$R\subseteq R \tag{11}$$

Transitivity

$$R\subseteq S \wedge S\subseteq T \quad\Rightarrow\quad R\subseteq T \tag{12}$$

Top and bottom

$$\bot \;\subseteq\; R \;\subseteq\; \top \tag{13}$$

Absorption

$$R\cdot\bot = \bot\cdot R = \bot \tag{14}$$

Relation equality

Pointwise

$$R = S \quad \textit{iff} \quad \langle \forall\, a, b \;:\; a \in A \wedge b \in B :\; b\, R\, a \;\Leftrightarrow\; b\, S\, a \rangle \tag{15}$$

“Ping-pong”

$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R \tag{16}$$

Indirect equality

$$\begin{aligned} R = S \quad &\equiv \quad \langle \forall\, X \;::\; (X \subseteq R \;\Leftrightarrow\; X \subseteq S) \rangle &(17)\\ &\equiv \quad \langle \forall\, X \;::\; (R \subseteq X \;\Leftrightarrow\; S \subseteq X) \rangle &(18) \end{aligned}$$

Typical layouts of indirect equality proofs

$$\left\{ \begin{array}{ll} & X \subseteq R \\ \equiv & \quad \{ \text{ ... } \} \\ & X \subseteq \ldots \\ \equiv & \quad \{ \text{ ... } \} \\ & X \subseteq S \\ :: & \quad \{ \text{ indirect equality } \} \\ & R = S \\ \Box & \end{array} \right. \qquad \left\{ \begin{array}{ll} & R \subseteq X \\ \equiv & \quad \{ \text{ ... } \} \\ & \ldots \subseteq X \\ \equiv & \quad \{ \text{ ... } \} \\ & S \subseteq X \\ :: & \quad \{ \text{ indirect equality } \} \\ & R = S \\ \Box & \end{array} \right.$$

Relation taxonomy

Kernel

$$\mathsf{ker}\ R = R^{\circ} \cdot R \tag{19}$$

Image

$$\mathsf{img}\ R \stackrel{\text{def}}{=} R \cdot R^{\circ} \tag{20}$$

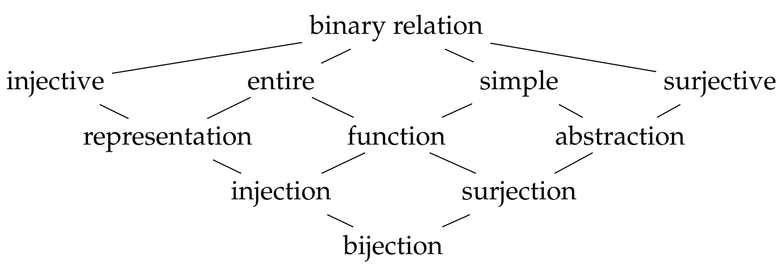
Duality

$$\begin{aligned} \mathsf{ker}\ (R^{\circ}) \quad &= \quad \mathsf{img}\ R &(21)\\ \mathsf{img}\ (R^{\circ}) \quad &= \quad \mathsf{ker}\ R &(22) \end{aligned}$$

Criteria

	Reflexive	*Coreflexive*
$\mathsf{ker}\ R$	entire R	injective R
$\mathsf{img}\ R$	surjective R	simple R

(23)



Taxonomy using matrices

- **entire** — at least one 1 in every column
- **surjective** — at least one 1 in every row
- **simple** — at most one 1 in every column
- **injective** — at most one 1 in every row
- **bijective** — exactly one 1 in evey column and every row.

Difunctional R

$$R \cdot R^{\circ} \cdot R \ \subseteq \ R \tag{24}$$

Functions

Shunting rules

$$f \cdot R \subseteq S \quad \equiv \quad R \subseteq f^{\circ} \cdot S \tag{25}$$
$$R \cdot f^{\circ} \subseteq S \quad \equiv \quad R \subseteq S \cdot f \tag{26}$$

Equality

$$f \subseteq g \ \equiv \ f = g \ \equiv \ f \supseteq g \tag{27}$$

Constant functions

Natural property

$$\underline{k} \cdot R \ \subseteq \ \underline{k} \tag{28}$$

Corollary

$$\mathsf{ker}\ ! = \frac{!}{!} = \top \tag{29}$$

Truth functions

$$true = \underline{\textsc{True}} \tag{30}$$
$$false = \underline{\textsc{False}} \tag{31}$$

Function division

Definition

$$\frac{f}{g} \quad = \quad g^{\circ} \cdot f \qquad cf. \quad \begin{tikzcd} B \arrow[dr, "g"'] & & A \arrow[ll, "\frac{f}{g}"'] \arrow[dl, "f"] \\ & C & \end{tikzcd} \tag{32}$$

Properties

$$\frac{f}{id} \quad = \quad f \tag{33}$$
$$\left(\frac{f}{g}\right)^{\circ} \quad = \quad \frac{g}{f} \tag{34}$$

$$\frac{f \cdot h}{g \cdot k} = k^\circ \cdot \frac{f}{g} \cdot h \tag{35}$$

$$\frac{f}{f} = \mathsf{ker}\, f \tag{36}$$

$$a \neq b \Leftrightarrow \frac{\underline{a}}{\underline{b}} = \bot \tag{37}$$

RELATION UNION

Pointwise definition

$$b\,(R \cup S)\,a \equiv b\,R\,a \vee b\,S\,a \tag{38}$$

Universal property

$$R \cup S \subseteq X \equiv R \subseteq X \wedge S \subseteq X \tag{39}$$

Right linearity

$$R \cdot (S \cup T) = (R \cdot S) \cup (R \cdot T) \tag{40}$$

Left linearity

$$(S \cup T) \cdot R = (S \cdot R) \cup (T \cdot R) \tag{41}$$

Converse- $\cup$

$$(R \cup S)^\circ = R^\circ \cup S^\circ \tag{42}$$

Top- $\cup$

$$\tag{43}$$

$$R \cup \top = \top \tag{44}$$

Bottom- $\cup$

$$\tag{45}$$

$$R \cup \bot = R \tag{46}$$

Union simplicity

$$M \cup N \text{ is simple} \equiv M, N \text{ are simple and } M \cdot N^\circ \subseteq id \tag{47}$$

RELATION INTERSECTION

Pointwise definition

$$b\,(R \cap S)\,a \equiv b\,R\,a \wedge b\,S\,a \tag{48}$$

Universal property

$$X \subseteq R \cap S \equiv X \subseteq R \wedge X \subseteq S \tag{49}$$

Distribution (1)

$$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \Leftarrow \left(\begin{array}{c} Q \cdot \mathsf{img}\, R \subseteq Q \\ \vee \\ S \cdot \mathsf{img}\, R \subseteq S \end{array} \right) \tag{50}$$

Distribution (2)

$$R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \quad \Leftarrow \quad \left(\begin{array}{c} (\mathsf{ker}\ R) \cdot Q \subseteq Q \\ \vee \\ (\mathsf{ker}\ R) \cdot S \subseteq S \end{array} \right) \tag{51}$$

Distribution (3)

$$g^\circ \cdot (R \cap S) \cdot f \quad = \quad g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f \tag{52}$$

Converse-$\cap$

$$(R \cap S)^\circ = R^\circ \cap S^\circ \tag{53}$$

Bottom-$\cap$

$$R \cap \bot = \bot \tag{54}$$

Top-$\cap$

$$R \cap \top = R \tag{55}$$

Misc.

$$k^\circ \cdot (f \cup g) = \frac{f}{k} \cup \frac{g}{k} \quad , \quad k^\circ \cdot (f \cap g) = \frac{f}{k} \cap \frac{g}{k} \tag{56}$$

Relation division

Universal-$/$

$$Z \cdot Y \subseteq X \ \equiv \ Z \subseteq X/Y \tag{57}$$

Pointwise-$/$

$$c\ (X/Y)\ a \ \equiv \ \langle \forall\, b \ : \ a\, Y\, b : \ c\, X\, b \rangle \tag{58}$$

$$\begin{tikzcd} & a \arrow[dl, "X/Y"'] \\ c & b \arrow[l, "X"] \arrow[u, "Y"'] \end{tikzcd}$$

Universal-$\backslash$

$$X \cdot Z \subseteq Y \ \Leftrightarrow \ Z \subseteq X \setminus Y \tag{59}$$

Pointwise-$\backslash$

$$a(X \setminus Y)c \quad \equiv \quad \langle \forall\, b \ : \ b\, X\, a : \ b\, Y\, c \rangle \tag{60}$$

Duality

$$(X \setminus Y)^\circ \quad = \quad Y^\circ / X^\circ \tag{61}$$

Misc.

$$X \cdot f \quad = \quad X / f^\circ \tag{62}$$

$$f \setminus X \quad = \quad f^\circ \cdot X \tag{63}$$

$$X / \bot \quad = \quad \top \tag{64}$$

$$X / id \quad = \quad X \tag{65}$$

$$R \setminus (f^\circ \cdot S) \quad = \quad f \cdot R \setminus S \tag{66}$$

$$R \setminus \top \cdot S \quad = \quad ! \cdot R \setminus ! \cdot S \tag{67}$$

$$R \,/\, (S \cup P) \quad = \quad R \,/\, S \cap R \,/\, P \tag{68}$$

Relation difference, implication and negation

Universal-$(-)$

$$X - R \subseteq Y \quad \equiv \quad X \subseteq Y \cup R \tag{69}$$

Pointwise-$\Rightarrow$

$$b(R \Rightarrow S)a \quad \equiv \quad (b\ R\ a) \Rightarrow (b\ S\ a) \tag{70}$$

Universal-$\Rightarrow$

$$R \cap X \subseteq Y \quad \equiv \quad X \subseteq (R \Rightarrow Y) \tag{71}$$

Distribution

$$f^\circ \cdot (R \Rightarrow S) \cdot g \quad = \quad (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g) \tag{72}$$

Definition-$\neg$

$$\neg R = (R \Rightarrow \bot) \tag{73}$$

Pointwise-$\neg$

$$b\ (\neg R)\ a \ \Leftrightarrow\ \neg\ (b\ R\ a) \tag{74}$$

Complementation

$$R \cup \neg R = \top \tag{75}$$

Difference versus implication

$$\top - R \ \subseteq\ R \Rightarrow \bot \tag{76}$$

de Morgan

$$\neg(R \cup S) = (\neg R) \cap (\neg S) \tag{77}$$

Schröder's rule

$$\neg Q \cdot S^\circ \ \subseteq\ \neg R \ \Leftrightarrow\ R^\circ \cdot \neg Q \ \subseteq\ \neg S \tag{78}$$

Relation shrinking

Definition-shrink

$$R \upharpoonright S \ = \ R \cap S / R^\circ \tag{79}$$

Universal-shrink

$$X \subseteq R \upharpoonright S \quad \equiv \quad X \subseteq R \ \wedge\ X \cdot R^\circ \subseteq S \tag{80}$$

Monotonicity

Composition

$$R \ \subseteq\ S \wedge T \ \subseteq\ U \quad \Rightarrow \quad R \cdot T \subseteq S \cdot U \tag{81}$$

Converse

$$R \ \subseteq\ S \quad \Rightarrow \quad R^\circ \ \subseteq\ S^\circ \tag{82}$$

Intersection

$$R \subseteq S \wedge U \subseteq V \quad \Rightarrow \quad R \cap U \subseteq S \cap V \tag{83}$$

Union

$$R \subseteq S \wedge U \subseteq V \quad \Rightarrow \quad R \cup U \subseteq S \cup V \tag{84}$$

Endo-relation taxonomy

Reflexive

$$id \subseteq R \tag{85}$$

Coreflexive

$$R \subseteq id \tag{86}$$

Transitive

$$R \cdot R \subseteq R \tag{87}$$

Symmetric

$$R \subseteq R^{\circ} (\equiv R = R^{\circ}) \tag{88}$$

Anti-symmetric

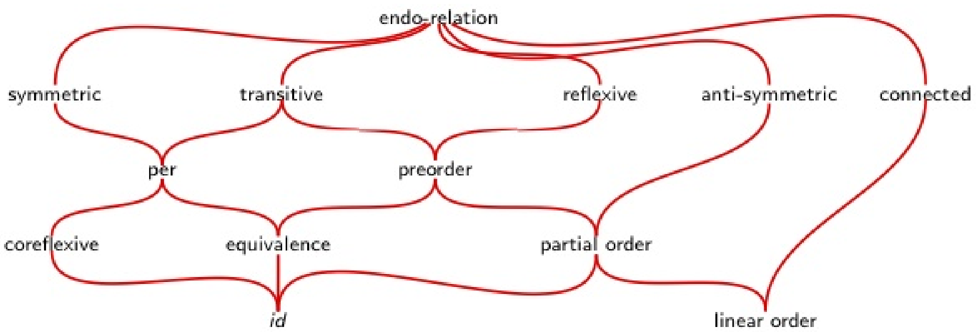
$$R \cap R^{\circ} \subseteq id \tag{89}$$

Irreflexive

$$R \cap id = \perp \tag{90}$$

Connected

$$R \cup R^{\circ} = \top \tag{91}$$



Coreflexives

Def-coreflexive

$$\Phi_p = id \cap \frac{true}{p} \tag{92}$$

Conv-coreflexive

$$\Phi_p^{\circ} = \Phi_p \tag{93}$$

Cond-coreflexive

$$\Phi_p \cdot \top = \frac{true}{p} \tag{94}$$

Pre-restriction

$$R \cdot \Phi_p \quad = \quad R \cap \top \cdot \Phi_p \tag{95}$$

Post-restriction

$$\Phi_q \cdot R \quad = \quad R \cap \Phi_q \cdot \top \tag{96}$$

Coreflexive-meet

$$\Phi_q \cdot \Phi_p = \Phi_q \cap \Phi_p \tag{97}$$

Guards

$$p? = [\Phi_p\,, \Phi_{\neg\, p}]^\circ \tag{98}$$

McCarthy

$$p \rightarrow R, S = [R\,, S] \cdot p? \tag{99}$$

Domain

$$\delta\, R = \mathsf{ker}\ R \cap id = \top \cdot R \cap id \tag{100}$$

Range

$$\rho\, R = \delta\, (R^\circ) \tag{101}$$

Dom-comp

$$\delta\, (R \cdot S) = \delta\, (\delta\, R \cdot S) \tag{102}$$

Comp-dom

$$R = R \cdot \delta\, R \tag{103}$$

Pairing ("splits")

Pairing-pointwise

$$(a, b)\ \langle R, S \rangle\ c \ \Leftrightarrow\ a\, R\, c \wedge b\, S\, c \tag{104}$$

Pairing-def

$$\langle R, S \rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S \tag{105}$$

Universal-pairing

$$X \subseteq \langle R, S \rangle \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{array} \right. \tag{106}$$

Pairing-fusion

$$\begin{array}{l} \langle R, S \rangle \cdot T = \langle R \cdot T, S \cdot T \rangle \\ \qquad \Leftarrow\ R \cdot (\mathsf{img}\ T) \subseteq R \vee S \cdot (\mathsf{img}\ T) \subseteq S \end{array} \tag{107}$$

Pairing-fusion (functions)

$$\langle R, S \rangle \cdot f \ = \ \langle R \cdot f, S \cdot f \rangle \tag{108}$$

Pairing and converse

$$\langle R, S \rangle^\circ \cdot \langle X, Y \rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y) \tag{109}$$

Kernel of pairing

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$$\mathsf{ker}\ \langle R, S \rangle \ = \ \mathsf{ker}\ R \cap \mathsf{ker}\ S \tag{111}$$

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Functor-$\times$ def

$$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2 \rangle \tag{113}$$

Functor-$\times$-id

$$id \times id = id \tag{114}$$

Functor-$\times$-composition

$$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q) \tag{115}$$

Functor-$\times$ absorption

$$(R \times S) \cdot \langle P, Q \rangle = \langle R \cdot P, S \cdot Q \rangle \tag{116}$$

Functor-$\times$-division

$$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k} \tag{117}$$

Pairing division

$$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f, h \rangle}{\langle g, k \rangle} \tag{118}$$

Tabulation

$$\pi_1 \cdot \pi_2^\circ = \top \tag{119}$$

Coproducts

Universal

$$X = [R, S] \quad \Leftrightarrow \quad \left\{ \begin{array}{l} X \cdot i_1 = R \\ X \cdot i_2 = S \end{array} \right. \tag{120}$$

Definition

$$[R, S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ \tag{121}$$

Reflexion

$$\mathsf{img}\ i_1 \cup \mathsf{img}\ i_2 = id \tag{122}$$

Disjointness

$$i_1^\circ \cdot i_2 = \bot \tag{123}$$

Divide & conquer

$$[R, S] \cdot [T, U]^\circ \quad = \quad (R \cdot T^\circ) \cup (S \cdot U^\circ) \tag{124}$$

Image of either

$$\mathsf{img}\ [R, S] \quad = \quad \mathsf{img}\ R \cup \mathsf{img}\ S \tag{125}$$

Exchange law

$$[\langle R, S \rangle, \langle T, V \rangle] \quad = \quad \langle [R, T], [S, V] \rangle \tag{126}$$

Functor-$+$-def

$$R + S = [i_1 \cdot R, i_2 \cdot S] \tag{127}$$

Functor-$+$-converse

$$(R + S)^\circ = R^\circ + S^\circ \tag{128}$$

Functor-$+$-division

$$\frac{f}{g} + \frac{h}{k} = \frac{f + h}{g + k} \tag{129}$$

INJECTIVITY PREORDER

Definition

$$R \leqslant S \quad \equiv \quad \mathsf{ker}\ S \ \subseteq \ \mathsf{ker}\ R \tag{130}$$

Join

$$\langle R, S \rangle \leqslant X \quad \equiv \quad R \leqslant X \wedge S \leqslant X \tag{131}$$

Cancellation

$$R \leqslant \langle R, S \rangle \ \text{ and } \ S \leqslant \langle R, S \rangle. \tag{132}$$

Shunting

$$R \cdot g \leqslant S \quad \equiv \quad R \leqslant S \cdot g^{\circ} \tag{133}$$

Meet

$$X \leqslant [R^{\circ}, S^{\circ}]^{\circ} \quad \Leftrightarrow \quad X \leqslant R \wedge X \leqslant S \tag{134}$$

Either-injective

$$id \leqslant [R, S] \ \Leftrightarrow \ id \leqslant R \wedge id \leqslant S \wedge R^{\circ} \cdot S = \bot \tag{135}$$

THUMB RULES

A function f is a bijection iff *its converse f° is a function g* (136)

- converse of *injective* is *simple* (and vice-versa) (137)

- converse of *entire* is *surjective* (and vice-versa) (138)

- *smaller* than injective (simple) is injective (simple) (139)

- *larger* than entire (surjective) is entire (surjective) (140)

$\langle R, id \rangle$ is always *injective*, for whatever R (141)

GALOIS CONNECTIONS

Summary

$(f\ b) \leqslant a \equiv b \sqsubseteq (g\ a)$		
Description	$f = g^{\flat}$	$g = f^{\sharp}$
Definition	$f\ b = \bigwedge\{a \mid b \sqsubseteq g\ a\}$	$g\ a = \bigsqcup\{b \mid f\ b \leqslant a\}$
Cancellation	$f(g\ a) \leqslant a$	$b \sqsubseteq g(f\ b)$
Distribution	$f(b \sqcup b') = (f\ b) \vee (f\ b')$	$g(a' \wedge a) = (g\ a') \sqcap (g\ a)$
Monotonicity	$b \sqsubseteq b' \Rightarrow f\ b \leqslant f\ b'$	$a \leqslant a' \Rightarrow g\ a \sqsubseteq g\ a'$

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Trading-∀

$$\langle \forall\, k \,:\, R \wedge S :\, T \rangle \quad = \quad \langle \forall\, k \,:\, R :\, S \Rightarrow T \rangle \tag{143}$$

Trading-∃

$$\langle \exists\, k \,:\, R \wedge S :\, T \rangle \quad = \quad \langle \exists\, k \,:\, R :\, S \wedge T \rangle \tag{144}$$

de Morgan

$$\neg \langle \forall\, k \,:\, R :\, T \rangle \quad = \quad \langle \exists\, k \,:\, R :\, \neg T \rangle \tag{145}$$

de Morgan

$$\neg \langle \exists\, k \,:\, R :\, T \rangle \quad = \quad \langle \forall\, k \,:\, R :\, \neg T \rangle \tag{146}$$

One-point-∀

$$\langle \forall\, k \,:\, k = e :\, T \rangle \quad = \quad T[k := e] \tag{147}$$

One-point-∃

$$\langle \exists\, k \,:\, k = e :\, T \rangle \quad = \quad T[k := e] \tag{148}$$

Nesting-∀

$$\langle \forall\, a, b \,:\, R \wedge S :\, T \rangle \quad = \quad \langle \forall\, a \,:\, R :\, \langle \forall\, b \,:\, S :\, T \rangle \rangle \tag{149}$$

Nesting-∃

$$\langle \exists\, a, b \,:\, R \wedge S :\, T \rangle \quad = \quad \langle \exists\, a \,:\, R :\, \langle \exists\, b \,:\, S :\, T \rangle \rangle \tag{150}$$

Rearranging-∀

$$\langle \forall\, k \,:\, R \vee S :\, T \rangle \quad = \quad \langle \forall\, k \,:\, R :\, T \rangle \wedge \langle \forall\, k \,:\, S :\, T \rangle \tag{151}$$

Rearranging-∀

$$\langle \forall\, k \,:\, R :\, T \wedge S \rangle \quad = \quad \langle \forall\, k \,:\, R :\, T \rangle \wedge \langle \forall\, k \,:\, R :\, S \rangle \tag{152}$$

Rearranging-∃

$$\langle \exists\, k \,:\, R :\, T \vee S \rangle \quad = \quad \langle \exists\, k \,:\, R :\, T \rangle \vee \langle \exists\, k \,:\, R :\, S \rangle \tag{153}$$

Rearranging-∃

$$\langle \exists\, k \,:\, R \vee S :\, T \rangle \quad = \quad \langle \exists\, k \,:\, R :\, T \rangle \vee \langle \exists\, k \,:\, S :\, T \rangle \tag{154}$$

Splitting-∀

$$\langle \forall\, j \,:\, R :\, \langle \forall\, k \,:\, S :\, T \rangle \rangle \quad = \quad \langle \forall\, k \,:\, \langle \exists\, j \,:\, R :\, S \rangle :\, T \rangle \tag{155}$$

Splitting-∃

$$\langle \exists\, j \,:\, R :\, \langle \exists\, k \,:\, S :\, T \rangle \rangle \quad = \quad \langle \exists\, k \,:\, \langle \exists\, j \,:\, R :\, S \rangle :\, T \rangle \tag{156}$$