# MFP/CSI/2425 — RELATION CALCULUS REFERENCE SHEET

## RELATIONAL COMPOSITION

**Pointwise def.**

$$\begin{tikzcd} B & A \arrow[l, "R"'] & C \arrow[l, "S"'] \arrow[ll, bend left, "R\cdot S"] \end{tikzcd} \quad b(R\cdot S)c \;\equiv\; \langle \exists\, a \;:\; b\,R\,a :\; a\,S\,c\rangle \tag{5.11}$$

**Associativity**

$$R\cdot(S\cdot P) = (R\cdot S)\cdot P \tag{5.12}$$

**Identity**

$$\left\{\begin{array}{l} R = R\cdot id_A \\ R = id_B\cdot R \end{array}\right. \tag{5.13}$$

## CONVERSE

**Pointwise def.**

$$b\,R\,a \quad\Leftrightarrow\quad a\,R^\circ\,b \tag{5.14}$$

**Universal-$^\circ$**

$$X^\circ \subseteq Y \quad\equiv\quad X \subseteq Y^\circ \tag{5.136}$$

**Involution**

$$(R^\circ)^\circ = R \tag{5.15}$$

**Contravariance**

$$(R\cdot S)^\circ = S^\circ\cdot R^\circ \tag{5.16}$$

**Isomorphism**

$$R\subseteq S \equiv R^\circ \subseteq S^\circ \tag{5.137}$$

**"Guardanapo"**

$$b(f^\circ\cdot R\cdot g)a \quad\equiv\quad (f\,b)R(g\,a) \tag{5.17}$$

## RELATION INCLUSION

**Pointwise**

$$R\subseteq S \quad \mathit{iff} \quad \langle \forall\, a,b \;::\; b\,R\,a \Rightarrow b\,S\,a\rangle \tag{5.19}$$

**Reflexion**

$$R\subseteq R \tag{5.21}$$

**Transitivity**

$$R\subseteq S \wedge S\subseteq T \quad\Rightarrow\quad R\subseteq T \tag{5.22}$$

**Top and bottom**

$$\bot \;\subseteq\; R \;\subseteq\; \top \tag{5.25}$$

**Absorption**

$$R\cdot\bot = \bot\cdot R = \bot \tag{5.26}$$

## RELATION EQUALITY

**Pointwise**

$$R = S \quad \mathit{iff} \quad \langle \forall\, a,b \;:\; a\in A\wedge b\in B :\; b\,R\,a \Leftrightarrow b\,S\,a\rangle \tag{5.18}$$

**"Ping-pong"**

$$R = S \quad\equiv\quad R\subseteq S\wedge S\subseteq R \tag{5.20}$$

**Indirect equality**

$$\begin{array}{rcl} R = S & \equiv & \langle \forall\, X \;::\; (X\subseteq R \Leftrightarrow X\subseteq S)\rangle \\ & \equiv & \langle \forall\, X \;::\; (R\subseteq X \Leftrightarrow S\subseteq X)\rangle \end{array} \tag{5.24}$$

**Typical layouts of indirect equality proofs**

$$\left\{\begin{array}{ll} & X \subseteq R \\ \equiv & \qquad \{ \ldots \} \\ & X \subseteq \ldots \\ \equiv & \qquad \{ \ldots \} \\ & X \subseteq S \\ :: & \qquad \{ \text{ indirect equality } \} \\ & R = S \\ \Box & \end{array}\right. \qquad \left\{\begin{array}{ll} & R \subseteq X \\ \equiv & \qquad \{ \ldots \} \\ & \ldots \subseteq X \\ \equiv & \qquad \{ \ldots \} \\ & S \subseteq X \\ :: & \qquad \{ \text{ indirect equality } \} \\ & R = S \\ \Box & \end{array}\right. \tag{5.24}$$

## RELATION TAXONOMY

**Kernel**

$$\ker R = R^{\circ} \cdot R \tag{5.32}$$

**Image**

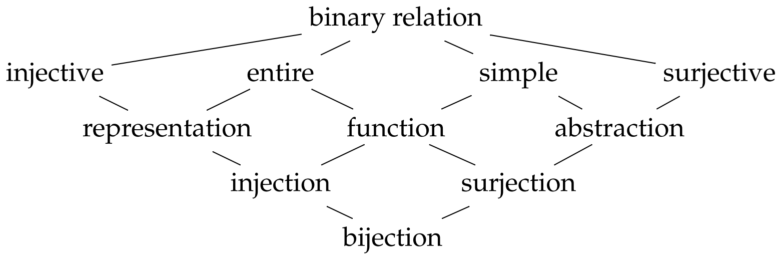
$$\mathsf{img}\ R \stackrel{\mathrm{def}}{=} R \cdot R^{\circ} \tag{5.33}$$

**Duality**

$$\begin{array}{rcl} \ker\ (R^{\circ}) & = & \mathsf{img}\ R \\ \mathsf{img}\ (R^{\circ}) & = & \ker\ R \end{array} \tag{5.34, 5.35}$$

**Criteria** (5.36)

| | *Reflexive* | *Coreflexive* |
|---|---|---|
| ker $R$ | entire $R$ | injective $R$ |
| img $R$ | surjective $R$ | simple $R$ |



### Taxonomy using matrices

- **entire** — at least one 1 in every column
- **surjective** — at least one 1 in every row
- **simple** — at most one 1 in every column
- **injective** — at most one 1 in every row
- **bijective** — exactly one 1 in evey column and every row.

**Difunctional** $R$

$$R \cdot R^{\circ} \cdot R \subseteq R \tag{§5.22}$$

## FUNCTIONS

**Shunting rules**

$$\begin{array}{rcl} f \cdot R \subseteq S & \equiv & R \subseteq f^{\circ} \cdot S \\ R \cdot f^{\circ} \subseteq S & \equiv & R \subseteq S \cdot f \end{array} \tag{5.46, 5.47}$$

**Equality**

$$f \subseteq g \equiv f = g \equiv f \supseteq g \tag{5.48}$$

## CONSTANT FUNCTIONS

**Natural property**

$$\underline{k} \cdot R \;\subseteq\; \underline{k} \tag{5.39}$$

**Corollary**

$$\mathsf{ker}\ ! = \frac{!}{!} = \top \tag{5.—}$$

**Truth functions**

$$\begin{array}{c} true = \underline{\textsc{True}} \\ false = \underline{\textsc{False}} \end{array} \tag{5.40, 5.41}$$

## FUNCTION DIVISION

**Definition**

$$\frac{f}{g} \;=\; g^{\circ} \cdot f \qquad cf. \quad \begin{tikzcd} B \arrow[dr, "g"'] & & A \arrow[ll, "\frac{f}{g}"'] \arrow[dl, "f"] \\ & C & \end{tikzcd} \tag{5.49}$$

**Properties**

$$\begin{array}{rcl} \frac{f}{id} & = & f \\ \left(\frac{f}{g}\right)^{\circ} & = & \frac{g}{f} \\ \frac{f \cdot h}{g \cdot k} & = & k^{\circ} \cdot \frac{f}{g} \cdot h \\ \frac{f}{f} & = & \mathsf{ker}\, f \\ a \neq b & \Leftrightarrow & \frac{\underline{a}}{\underline{b}} = \bot \end{array} \tag{5.50 → 5.54}$$

## RELATION UNION

**Pointwise definition**

$$b\,(R \cup S)\,a \;\;\equiv\;\; b\,R\,a \vee b\,S\,a \tag{5.57}$$

**Universal property**

$$R \cup S \;\subseteq\; X \;\;\equiv\;\; R \subseteq X \;\wedge\; S \subseteq X \tag{5.59}$$

**Right linearity**

$$R \cdot (S \cup T) \;\;=\;\; (R \cdot S) \cup (R \cdot T) \tag{5.60}$$

**Left linearity**

$$(S \cup T) \cdot R \;\;=\;\; (S \cdot R) \cup (T \cdot R) \tag{5.61}$$

**Converse-** $\cup$

$$(R \cup S)^{\circ} = R^{\circ} \cup S^{\circ} \tag{5.65}$$

**Top-** $\cup$

$$R \cup \top = \top \tag{5.68}$$

**Bottom-** $\cup$

$$R \cup \bot = R \tag{5.69}$$

**Union simplicity**

$$M \cup N \text{ is simple} \;\;\equiv\;\; M, N \text{ are simple and } M \cdot N^{\circ} \subseteq id \tag{5.70}$$

## RELATION INTERSECTION

**Pointwise definition**

$$b\,(R \cap S)\,a \;\;\equiv\;\; b\,R\,a \wedge b\,S\,a \tag{5.56}$$

**Universal property**

$$X \;\subseteq\; R \cap S \;\;\equiv\;\; X \subseteq R \;\wedge\; X \subseteq S \tag{5.58}$$

**Distribution (1)**

$$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \;\;\Leftarrow\;\; \left( \begin{array}{c} Q \cdot \mathsf{img}\ R \subseteq Q \\ \vee \\ S \cdot \mathsf{img}\ R \subseteq S \end{array} \right) \tag{5.62}$$

**Distribution (2)**

$$R\cdot(Q\cap S)=(R\cdot Q)\cap(R\cdot S)\quad\Leftarrow\quad\left(\begin{array}{c}(\ker R)\cdot Q\subseteq Q\\ \vee\\ (\ker R)\cdot S\subseteq S\end{array}\right)\tag{5.63}$$

**Distribution (3)**

$$g^\circ\cdot(R\cap S)\cdot f\quad=\quad g^\circ\cdot R\cdot f\cap g^\circ\cdot S\cdot f\tag{5.71}$$

**Converse-$\cap$**

$$(R\cap S)^\circ=R^\circ\cap S^\circ\tag{5.64}$$

**Bottom-$\cap$**

$$R\cap\bot=\bot\tag{5.66}$$

**Top-$\cap$**

$$R\cap\top=R\tag{5.67}$$

**Misc.**

$$k^\circ\cdot(f\cup g)=\tfrac{f}{k}\cup\tfrac{g}{k}\quad,\quad k^\circ\cdot(f\cap g)=\tfrac{f}{k}\cap\tfrac{g}{k}\tag{5.73}$$

## RELATION DIVISION

**Universal-/**

$$Z\cdot Y\subseteq X\;\equiv\;Z\subseteq X/Y\tag{5.157}$$

**Pointwise-/**

$$c\,(X/Y)\,a\;\equiv\;\langle\forall\,b\;:\;a\,Y\,b:\;c\,X\,b\rangle\tag{5.158}$$

$$\begin{tikzcd} & a \arrow[dl, "X/Y"'] \\ c & b \arrow[l, "X"] \arrow[u, "Y"'] \end{tikzcd}$$

**Universal-$\setminus$**

$$X\cdot Z\subseteq Y\;\Leftrightarrow\;Z\subseteq X\setminus Y\tag{5.159}$$

**Pointwise-$\setminus$**

$$a(X\setminus Y)c\quad\equiv\quad\langle\forall\,b\;:\;b\,X\,a:\;b\,Y\,c\rangle\tag{5.160}$$

**Duality**

$$(X\setminus Y)^\circ\quad=\quad Y^\circ/X^\circ\tag{5.—}$$

**Misc.**

$$X\cdot f\quad=\quad X/f^\circ\tag{5.162}$$

$$f\setminus X\quad=\quad f^\circ\cdot X\tag{5.163}$$

$$X/\bot\quad=\quad\top\tag{5.164}$$

$$X/id\quad=\quad X\tag{5.165}$$

$$R\setminus(f^\circ\cdot S)\quad=\quad f\cdot R\setminus S\tag{5.166}$$

$$R\setminus\top\cdot S\quad=\quad!\cdot R\setminus!\cdot S\tag{5.167}$$

$$R\,/\,(S\cup P)\quad=\quad R\,/\,S\cap R\,/\,P\tag{5.168}$$

## RELATION DIFFERENCE, IMPLICATION AND NEGATION

**Universal-$(-)$**

$$X-R\subseteq Y\quad\equiv\quad X\subseteq Y\cup R\tag{5.138}$$

**Pointwise-$\Rightarrow$**

$$b(R\Rightarrow S)a\quad\equiv\quad(b\;R\;a)\Rightarrow(b\;S\;a)\tag{5.147}$$

**Universal-$\Rightarrow$**

$$R\cap X\subseteq Y\quad\equiv\quad X\subseteq(R\Rightarrow Y)\tag{5.148}$$

**Distribution**

$$f^\circ\cdot(R\Rightarrow S)\cdot g\quad=\quad(f^\circ\cdot R\cdot g)\Rightarrow(f^\circ\cdot S\cdot g)\tag{5.154}$$

| | | |
|---|---|---|
| **Definition-¬** | $\neg R = (R \Rightarrow \bot)$ | (5.—-) |
| **Pointwise-¬** | $b\ (\neg R)\ a \ \Leftrightarrow\ \neg\ (b\ R\ a)$ | (5.—-) |
| **Complementation** | $R \cup \neg R = \top$ | (5.152) |
| **Difference versus implication** | $\top - R\ \subseteq\ R \Rightarrow \bot$ | (5.153) |
| **de Morgan** | $\neg(R \cup S) = (\neg R) \cap (\neg S)$ | (5.154) |
| **Schröder's rule** | $\neg Q \cdot S^\circ\ \subseteq\ \neg R\ \Leftrightarrow\ R^\circ \cdot \neg Q\ \subseteq\ \neg S$ | (5.151) |

## RELATION SHRINKING

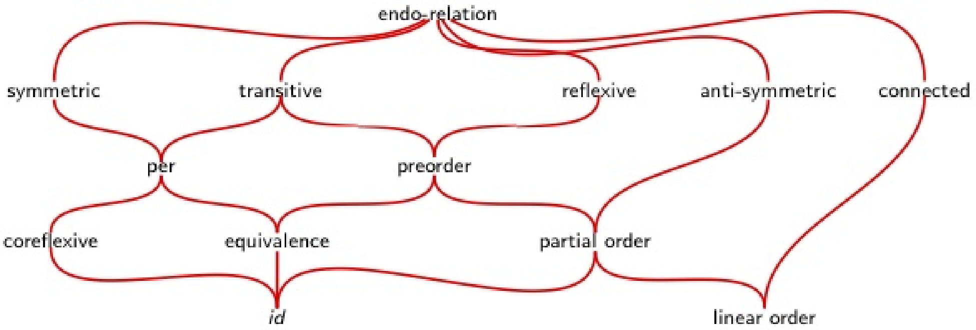
| | | |
|---|---|---|
| **Definition-shrink** | $R \upharpoonright S\ =\ R \cap S / R^\circ$ | (5.183) |
| **Universal-shrink** | $X \subseteq R \upharpoonright S\ \equiv\ X \subseteq R\ \wedge\ X \cdot R^\circ \subseteq S$ | (5.182) |

## MONOTONICITY

| | | |
|---|---|---|
| **Composition** | $R\ \subseteq\ S \wedge T\ \subseteq\ U\ \ \Rightarrow\ \ R \cdot T \subseteq S \cdot U$ | (5.78) |
| **Converse** | $R\ \subseteq\ S\ \ \Rightarrow\ \ R^\circ\ \subseteq\ S^\circ$ | (5.79) |
| **Intersection** | $R \subseteq S \wedge U \subseteq V\ \ \Rightarrow\ \ R \cap U \subseteq S \cap V$ | (5.80) |
| **Union** | $R \subseteq S \wedge U \subseteq V\ \ \Rightarrow\ \ R \cup U \subseteq S \cup V$ | (5.81) |

## ENDO-RELATION TAXONOMY

| | | |
|---|---|---|
| **Reflexive** | $id \subseteq R$ | (5.84) |
| **Coreflexive** | $R \subseteq id$ | (5.85) |
| **Transitive** | $R \cdot R \subseteq R$ | (5.86) |
| **Symmetric** | $R \subseteq R^\circ (\equiv R = R^\circ)$ | (5.87) |
| **Anti-symmetric** | $R \cap R^\circ\ \subseteq id$ | (5.88) |
| **Irreflexive** | $R \cap id = \bot$ | (5.89) |
| **Connected** | $R \cup R^\circ = \top$ | (5.90) |



## Coreflexives

| | | |
|---|---|---|
| **Def-coreflexive** | $\Phi_p = id \cap \frac{true}{p}$ | (5.202) |
| **Conv-coreflexive** | $\Phi_p^\circ = \Phi_p$ | (5.203) |
| **Cond-coreflexive** | $\Phi_p \cdot \top = \frac{true}{p}$ | (5.209) |
| **Pre-restriction** | $R \cdot \Phi_p \quad = \quad R \cap \top \cdot \Phi_p$ | (5.210) |
| **Post-restriction** | $\Phi_q \cdot R \quad = \quad R \cap \Phi_q \cdot \top$ | (5.211) |
| **Coreflexive-meet** | $\Phi_q \cdot \Phi_p = \Phi_q \cap \Phi_p$ | (2.212) |
| **Guards** | $p? = [\Phi_p , \Phi_{\neg\, p}]^\circ$ | (2.215) |
| **McCarthy** | $p \rightarrow R, S = [R, S] \cdot p?$ | (2.216) |
| **Domain** | $\delta\, R = \mathsf{ker}\, R \cap id = \top \cdot R \cap id$ | (2.218) |
| **Range** | $\rho\, R = \delta\, (R^\circ)$ | (2.220) |
| **Dom-comp** | $\delta\, (R \cdot S) = \delta\, (\delta\, R \cdot S)$ | (2.227) |
| **Comp-dom** | $R = R \cdot \delta\, R$ | (2.229) |

## Pairing (”splits”)

| | | |
|---|---|---|
| **Pairing-pointwise** | $(a, b)\ \langle R, S\rangle\ c \ \Leftrightarrow\ a\, R\, c \wedge b\, S\, c$ | (5.101) |
| **Pairing-def** | $\langle R, S\rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S$ | (5.102) |
| **Universal-pairing** | $X \subseteq \langle R, S\rangle \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{array} \right.$ | (5.103) |
| **Pairing-fusion** | $\langle R, S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle$<br>$\Leftarrow\ R \cdot (\mathsf{img}\ T) \subseteq R \vee S \cdot (\mathsf{img}\ T) \subseteq S$ | (5.104) |
| **Pairing-fusion (functions)** | $\langle R, S\rangle \cdot f \ = \ \langle R \cdot f, S \cdot f\rangle$ | (5.105) |
| **Pairing and converse** | $\langle R, S\rangle^\circ \cdot \langle X, Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y)$ | (5.108) |

| | | |
|---|---|---|
| **Kernel of pairing** | $\mathsf{ker}\ \langle R, S\rangle \ = \ \mathsf{ker}\ R \cap \mathsf{ker}\ S$ | (5.111) |
| **Functor-$\times$ def** | $R \times S = \langle R \cdot \pi_1, S \cdot \pi_2 \rangle$ | (5.107) |
| **Functor-$\times$-id** | $id \times id = id$ | (5.112) |
| **Functor-$\times$-composition** | $(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q)$ | (5.113) |
| **Functor-$\times$ absorption** | $(R \times S) \cdot \langle P, Q \rangle = \langle R \cdot P, S \cdot Q \rangle$ | (5.106) |
| **Functor-$\times$-division** | $\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k}$ | (5.127) |
| **Pairing division** | $\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f, h \rangle}{\langle g, k \rangle}$ | (5.109) |
| **Tabulation** | $\pi_1 \cdot \pi_2^\circ = \top$ | (5.—) |

## COPRODUCTS

| | | |
|---|---|---|
| **Universal** | $X = [R, S] \ \Leftrightarrow \ \left\{ \begin{array}{l} X \cdot i_1 = R \\ X \cdot i_2 = S \end{array} \right.$ | (5.114) |
| **Definition** | $[R, S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ$ | (5.117) |
| **Reflexion** | $\mathsf{img}\ i_1 \cup \mathsf{img}\ i_2 = id$ | (5.115) |
| **Disjointness** | $i_1^\circ \cdot i_2 = \bot$ | (5.116) |
| **Divide & conquer** | $[R, S] \cdot [T, U]^\circ \ = \ (R \cdot T^\circ) \cup (S \cdot U^\circ)$ | (5.121) |
| **Image of either** | $\mathsf{img}\ [R, S] \ = \ \mathsf{img}\ R \cup \mathsf{img}\ S$ | (5.124) |
| **Exchange law** | $[\langle R, S \rangle , \langle T, V \rangle] \ = \ \langle [R , T], [S , V] \rangle$ | (5.122) |
| **Functor-$+$-def** | $R + S = [i_1 \cdot R , i_2 \cdot S]$ | (5.119) |
| **Functor-$+$-converse** | $(R + S)^\circ = R^\circ + S^\circ$ | (5.123) |
| **Functor-$+$-division** | $\frac{f}{g} + \frac{h}{k} = \frac{f + h}{g + k}$ | (5.128) |

## INJECTIVITY PREORDER

| | | |
|---|---|---|
| **Definition** | $R \leqslant S \ \equiv \ \mathsf{ker}\ S \subseteq \mathsf{ker}\ R$ | (5.234) |
| **Join** | $\langle R, S \rangle \leqslant X \ \equiv \ R \leqslant X \wedge S \leqslant X$ | (5.235) |
| **Cancellation** | $R \leqslant \langle R, S \rangle$ and $S \leqslant \langle R, S \rangle.$ | (5.236) |
| **Shunting** | $R \cdot g \leqslant S \ \equiv \ R \leqslant S \cdot g^\circ$ | (5.237) |
| **Meet** | $X \leqslant [R^\circ , S^\circ]^\circ \ \Leftrightarrow \ X \leqslant R \wedge X \leqslant S$ | (5.238) |

**Either-injective** $$id \leqslant [R,S] \Leftrightarrow id \leqslant R \wedge id \leqslant S \wedge R^{\circ} \cdot S = \bot \tag{5.126}$$

## THUMB RULES

*A function $f$ is a bijection* iff *its converse $f^{\circ}$ is a function $g$* (5.38)

- converse of *injective* is *simple* (and vice-versa) (5.43)

- converse of *entire* is *surjective* (and vice-versa) (5.44)

- *smaller* than injective (simple) is injective (simple) (5.82)

- *larger* than entire (surjective) is entire (surjective) (5.83)

$\langle R, id \rangle$ is always *injective*, for whatever $R$ (5.111a)

## GALOIS CONNECTIONS

**Summary** (§5.18)

| $(f\ b) \leqslant a \equiv b \sqsubseteq (g\ a)$ | | |
|---|---|---|
| **Description** | $f = g^{\flat}$ | $g = f^{\sharp}$ |
| Definition | $f\ b = \bigwedge\{a \mid b \sqsubseteq g\ a\}$ | $g\ a = \bigsqcup\{b \mid f\ b \leqslant a\}$ |
| Cancellation | $f(g\ a) \leqslant a$ | $b \sqsubseteq g(f\ b)$ |
| Distribution | $f(b \sqcup b') = (f\ b) \vee (f\ b')$ | $g(a' \wedge a) = (g\ a') \sqcap (g\ a)$ |
| Monotonicity | $b \sqsubseteq b' \Rightarrow f\ b \leqslant f\ b'$ | $a \leqslant a' \Rightarrow g\ a \sqsubseteq g\ a'$ |

## QUANTIFIER CALCULUS

**Trading-∀** $$\langle \forall\, k \,:\, R \wedge S :\, T \rangle \;=\; \langle \forall\, k \,:\, R :\, S \Rightarrow T \rangle \tag{A.1}$$

**Trading-∃** $$\langle \exists\, k \,:\, R \wedge S :\, T \rangle \;=\; \langle \exists\, k \,:\, R :\, S \wedge T \rangle \tag{A.2}$$

**de Morgan** $$\neg \langle \forall\, k \,:\, R :\, T \rangle \;=\; \langle \exists\, k \,:\, R :\, \neg T \rangle \tag{A.3}$$

**de Morgan** $$\neg \langle \exists\, k \,:\, R :\, T \rangle \;=\; \langle \forall\, k \,:\, R :\, \neg T \rangle \tag{A.4}$$

**One-point-∀** $$\langle \forall\, k \,:\, k = e :\, T \rangle \;=\; T[k := e] \tag{A.5}$$

**One-point-∃** $$\langle \exists\, k \,:\, k = e :\, T \rangle \;=\; T[k := e] \tag{A.6}$$

**Nesting-∀** $$\langle \forall\, a, b \,:\, R \wedge S :\, T \rangle \;=\; \langle \forall\, a \,:\, R :\, \langle \forall\, b \,:\, S :\, T \rangle \rangle \tag{A.7}$$

**Nesting-∃** $$\langle \exists\, a, b \,:\, R \wedge S :\, T \rangle \;=\; \langle \exists\, a \,:\, R :\, \langle \exists\, b \,:\, S :\, T \rangle \rangle \tag{A.8}$$

**Rearranging-∀** $$\langle \forall\, k \,:\, R \vee S :\, T \rangle \;=\; \langle \forall\, k \,:\, R :\, T \rangle \wedge \langle \forall\, k \,:\, S :\, T \rangle \tag{A.9}$$

**Rearranging-∀** $$\langle \forall\, k \,:\, R :\, T \wedge S \rangle \;=\; \langle \forall\, k \,:\, R :\, T \rangle \wedge \langle \forall\, k \,:\, R :\, S \rangle \tag{A.10}$$

**Rearranging-∃** $$\langle \exists\, k \,:\, R :\, T \vee S \rangle \;=\; \langle \exists\, k \,:\, R :\, T \rangle \vee \langle \exists\, k \,:\, R :\, S \rangle \tag{A.11}$$

**Rearranging-∃** $$\langle \exists\, k \,:\, R \vee S :\, T \rangle \;=\; \langle \exists\, k \,:\, R :\, T \rangle \vee \langle \exists\, k \,:\, S :\, T \rangle \tag{A.12}$$

**Splitting-∀** $$\langle \forall\, j \,:\, R : \langle \forall\, k \,:\, S : T \rangle\rangle \quad = \quad \langle \forall\, k \,:\, \langle \exists\, j \,:\, R : S \rangle : T \rangle \tag{A.13}$$

**Splitting-∃** $$\langle \exists\, j \,:\, R : \langle \exists\, k \,:\, S : T \rangle\rangle \quad = \quad \langle \exists\, k \,:\, \langle \exists\, j \,:\, R : S \rangle : T \rangle \tag{A.14}$$