MFES/CSI/2324 — Relation Calculus Reference Sheet

Relational composition

Pointwise def.	$\begin{tikzcd} B & A \arrow[l, "R"'] & C \arrow[l, "S"'] \arrow[ll, bend left, "R\cdot S"] \end{tikzcd} \quad b(R\cdot S)c \ \equiv\ \langle \exists\, a \,:\, b\,R\,a :\, a\,S\,c\rangle$	(5.11)
Associativity	$R\cdot(S\cdot P) = (R\cdot S)\cdot P$	(5.12)
Identity	$\left\{ \begin{array}{l} R = R\cdot id_A \\ R = id_B\cdot R \end{array} \right.$	(5.13)

Converse

Pointwise def.	$b\,R\,a \quad\Leftrightarrow\quad a\,R^\circ\,b$	(5.14)
Universal-$^\circ$	$X^\circ \subseteq Y \quad\equiv\quad X \subseteq Y^\circ$	(5.136)
Involution	$(R^\circ)^\circ = R$	(5.15)
Contravariance	$(R\cdot S)^\circ = S^\circ\cdot R^\circ$	(5.16)
Isomorphism	$R\subseteq S \equiv R^\circ \subseteq S^\circ$	(5.137)
“Guardanapo”	$b(f^\circ\cdot R\cdot g)a \quad\equiv\quad (f\,b)R(g\,a)$	(5.17)

Relation inclusion

Pointwise	$R\subseteq S \quad \mathit{iff} \quad \langle\forall\, a,b \,::\, b\,R\,a \Rightarrow b\,S\,a\rangle$	(5.19)
Reflexion	$R\subseteq R$	(5.21)
Transitivity	$R\subseteq S \wedge S\subseteq T \quad\Rightarrow\quad R\subseteq T$	(5.22)
Top and bottom	$\bot \ \subseteq\ R\ \subseteq\ \top$	(5.25)
Absorption	$R\cdot\bot = \bot\cdot R = \bot$	(5.26)

Relation equality

Pointwise	$R = S \quad \mathit{iff} \quad \langle\forall\, a,b \,:\, a \in A \wedge b \in B :\, b\,R\,a \Leftrightarrow b\,S\,a\rangle$	(5.18)
Indirect equality	$\begin{array}{rcl} R = S & \equiv & \langle\forall\, X \,::\, (X \subseteq R \Leftrightarrow X \subseteq S)\rangle \\ & \equiv & \langle\forall\, X \,::\, (R \subseteq X \Leftrightarrow S \subseteq X)\rangle \end{array}$	(5.24)
“Ping-pong”	$R = S \quad\equiv\quad R\subseteq S \wedge S\subseteq R$	(5.20)

RELATION TAXONOMY

Kernel

$$\mathsf{ker}\ R = R^{\circ} \cdot R \tag{5.32}$$

Image

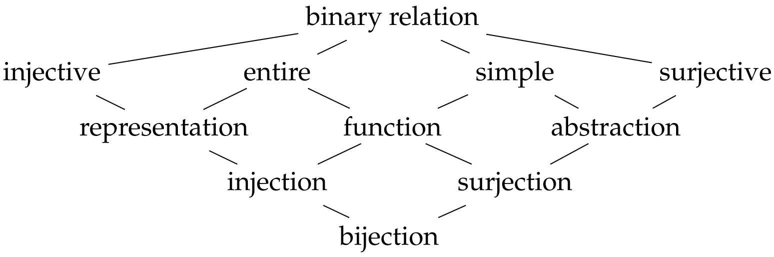
$$\mathsf{img}\ R \overset{\mathrm{def}}{=} R \cdot R^{\circ} \tag{5.33}$$

Duality

$$\begin{array}{rcl} \mathsf{ker}\ (R^{\circ}) & = & \mathsf{img}\ R \\ \mathsf{img}\ (R^{\circ}) & = & \mathsf{ker}\ R \end{array} \tag{5.34, 5.35}$$

Criteria (5.36)

	Reflexive	*Coreflexive*
$\mathsf{ker}\ R$	entire R	injective R
$\mathsf{img}\ R$	surjective R	simple R



Taxonomy using matrices

- **entire** — at least one 1 in every column
- **surjective** — at least one 1 in every row
- **simple** — at most one 1 in every column
- **injective** — at most one 1 in every row
- **bijective** — exactly one 1 in evey column and every row.

Difunctional R

$$R \cdot \mathsf{ker}\ R \ \subseteq\ R \tag{§5.22}$$

FUNCTIONS

Shunting rules

$$\begin{array}{rcl} f \cdot R \subseteq S & \equiv & R \subseteq f^{\circ} \cdot S \\ R \cdot f^{\circ} \subseteq S & \equiv & R \subseteq S \cdot f \end{array} \tag{5.46, 5.47}$$

Equality

$$f \subseteq g \ \equiv\ f = g \ \equiv\ f \supseteq g \tag{5.48}$$

CONSTANT FUNCTIONS

Natural property

$$\underline{k} \cdot R \ \subseteq\ \underline{k} \tag{5.39}$$

Corollary

$$\mathsf{ker}\ \ ! = \frac{!}{!} = \top \tag{5.—}$$

Truth functions

$$\begin{array}{c} \mathit{true} = \underline{\textsc{True}} \\ \mathit{false} = \underline{\textsc{False}} \end{array} \tag{5.40, 5.41}$$

FUNCTION DIVISION

Definition

$$\frac{f}{g} \;=\; g^\circ \cdot f \qquad cf. \quad \begin{tikzcd} B \arrow[dr, "g"'] & & A \arrow[ll, "\frac{f}{g}"'] \arrow[dl, "f"] \\ & C & \end{tikzcd}$$ (5.49)

Properties

$$\begin{array}{rcl} \frac{f}{id} & = & f \\ \left(\frac{f}{g}\right)^\circ & = & \frac{g}{f} \\ \frac{f\cdot h}{g\cdot k} & = & k^\circ \cdot \frac{f}{g} \cdot h \\ \frac{f}{f} & = & \mathsf{ker}\, f \\ a \neq b & \Leftrightarrow & \frac{\underline{a}}{\underline{b}} = \bot \end{array}$$ (5.50 → 5.54)

RELATION UNION

Pointwise definition $b\,(R \cup S)\,a \;\equiv\; b\,R\,a \vee b\,S\,a$ (5.57)

Universal property $R \cup S \subseteq X \;\equiv\; R \subseteq X \,\wedge\, S \subseteq X$ (5.59)

Right linearity $R \cdot (S \cup T) \;=\; (R \cdot S) \cup (R \cdot T)$ (5.60)

Left linearity $(S \cup T) \cdot R \;=\; (S \cdot R) \cup (T \cdot R)$ (5.61)

Converse- $\cup$ $(R \cup S)^\circ = R^\circ \cup S^\circ$ (5.65)

Top- $\cup$ $R \cup \top = \top$ (5.68)

Bottom- $\cup$ $R \cup \bot = R$ (5.69)

Union simplicity $M \cup N$ is simple $\equiv$ M, N are simple and $M \cdot N^\circ \subseteq id$ (5.70)

RELATION INTERSECTION

Pointwise definition $b\,(R \cap S)\,a \;\equiv\; b\,R\,a \wedge b\,S\,a$ (5.56)

Universal property $X \subseteq R \cap S \;\equiv\; X \subseteq R \,\wedge\, X \subseteq S$ (5.58)

Distribution (1)

$$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \;\Leftarrow\; \left\{ \begin{array}{c} Q \cdot \mathsf{img}\, R \subseteq Q \\ \vee \\ S \cdot \mathsf{img}\, R \subseteq S \end{array} \right.$$ (5.62)

Distribution (2)

$$R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \;\Leftarrow\; \left\{ \begin{array}{c} (\mathsf{ker}\, R) \cdot Q \subseteq Q \\ \vee \\ (\mathsf{ker}\, R) \cdot S \subseteq S \end{array} \right.$$ (5.63)

Distribution (3) $g^\circ \cdot (R \cap S) \cdot f \;=\; g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f$ (5.71)

Converse- $\cap$ $(R \cap S)^\circ = R^\circ \cap S^\circ$ (5.64)

Bottom- $\cap$ $R \cap \bot = \bot$ (5.66)

Top-$\cap$

$$R \cap \top = R \tag{5.67}$$

Misc.

$$k^\circ \cdot (f \cup g) = \tfrac{f}{k} \cup \tfrac{g}{k} \quad , \quad k^\circ \cdot (f \cap g) = \tfrac{f}{k} \cap \tfrac{g}{k} \tag{5.73}$$

RELATION DIVISION

Universal-$/$

$$Z \cdot Y \subseteq X \;\equiv\; Z \subseteq X/Y \tag{5.157}$$

Pointwise-$/$

$$c\;(X/Y)\;a \;\equiv\; \langle \forall\, b \;:\; a\,Y\,b :\; c\,X\,b \rangle \tag{5.158}$$

$$\begin{tikzcd}
 & a \arrow[dl, "X/Y"'] \\
c & b \arrow[l, "X"] \arrow[u, "Y"']
\end{tikzcd}$$

Universal-$\backslash$

$$X \cdot Z \subseteq Y \;\Leftrightarrow\; Z \subseteq X \setminus Y \tag{5.159}$$

Pointwise-$\backslash$

$$a(X \setminus Y)c \quad\equiv\quad \langle \forall\, b \;:\; b\,X\,a :\; b\,Y\,c \rangle \tag{5.160}$$

Misc.

$$X \cdot f \quad=\quad X/f^\circ \tag{5.162}$$

$$f \setminus X \quad=\quad f^\circ \cdot X \tag{5.163}$$

$$X/\bot \quad=\quad \top \tag{5.164}$$

$$X/id \quad=\quad X \tag{5.165}$$

$$R \setminus (f^\circ \cdot S) \quad=\quad f \cdot R \setminus S \tag{5.166}$$

$$R \setminus \top \cdot S \quad=\quad ! \cdot R \setminus ! \cdot S \tag{5.167}$$

$$R \;/\; (S \cup P) \quad=\quad R \;/\; S \cap R \;/\; P \tag{5.168}$$

RELATION DIFFERENCE, IMPLICATION AND NEGATION

Universal-$(-)$

$$X - R \subseteq Y \quad\equiv\quad X \subseteq Y \cup R \tag{5.138}$$

Pointwise-$\Rightarrow$

$$b(R \Rightarrow S)a \quad\equiv\quad (b\;R\;a) \Rightarrow (b\;S\;a) \tag{5.147}$$

Universal-$\Rightarrow$

$$R \cap X \subseteq Y \quad\equiv\quad X \subseteq (R \Rightarrow Y) \tag{5.148}$$

Distribution

$$f^\circ \cdot (R \Rightarrow S) \cdot g \quad=\quad (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g) \tag{5.154}$$

Definition-$\neg$

$$\neg R = (R \Rightarrow \bot) \tag*{(5.—–)}$$

Pointwise-$\neg$

$$b\;(\neg R)\;a \;\Leftrightarrow\; \neg\;(b\;R\;a) \tag*{(5.—–)}$$

Complementation

$$R \cup \neg R = \top \tag{5.152}$$

Difference versus implication

$$\top - R \;\subseteq\; R \Rightarrow \bot \tag{5.153}$$

de Morgan

$$\neg(R \cup S) = (\neg R) \cap (\neg S) \tag{5.154}$$

Schröder's rule

$$\neg Q \cdot S^\circ \;\subseteq\; \neg R \;\Leftrightarrow\; R^\circ \cdot \neg Q \;\subseteq\; \neg S \tag{5.151}$$

RELATION SHRINKING

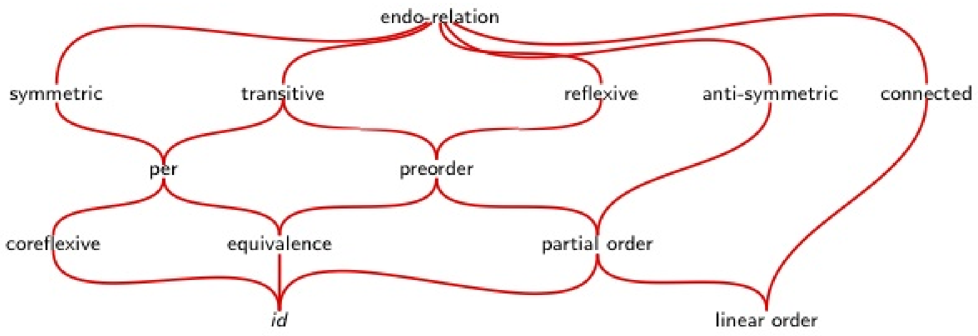
Definition-shrink	$R \upharpoonright S \;=\; R \cap S / R^\circ$	(5.183)
Universal-shrink	$X \subseteq R \upharpoonright S \;\equiv\; X \subseteq R \,\wedge\, X \cdot R^\circ \subseteq S$	(5.182)

MONOTONICITY

Composition	$R \;\subseteq\; S \wedge T \;\subseteq\; U \;\;\Rightarrow\;\; R \cdot T \subseteq S \cdot U$	(5.78)
Converse	$R \;\subseteq\; S \;\;\Rightarrow\;\; R^\circ \;\subseteq\; S^\circ$	(5.79)
Intersection	$R \subseteq S \wedge U \subseteq V \;\;\Rightarrow\;\; R \cap U \subseteq S \cap V$	(5.80)
Union	$R \subseteq S \wedge U \subseteq V \;\;\Rightarrow\;\; R \cup U \subseteq S \cup V$	(5.81)

ENDO-RELATION TAXONOMY

Reflexive	$id \subseteq R$	(5.84)
Coreflexive	$R \subseteq id$	(5.85)
Transitive	$R \cdot R \subseteq R$	(5.86)
Symmetric	$R \subseteq R^\circ (\equiv R = R^\circ)$	(5.87)
Anti-symmetric	$R \cap R^\circ \;\subseteq id$	(5.88)
Irreflexive	$R \cap id = \bot$	(5.89)
Connected	$R \cup R^\circ = \top$	(5.90)



PAIRING ("SPLITS")

Pairing-pointwise	$(a,b)\ \langle R,S \rangle\ c \;\Leftrightarrow\; a\, R\, c \wedge b\, S\, c$	(5.101)
Pairing-def	$\langle R,S \rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S$	(5.102)

Universal-pairing	$X \subseteq \langle R, S\rangle \quad \Leftrightarrow \quad \begin{cases} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{cases}$	(5.103)
Pairing-fusion	$\langle R, S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle$ $\Leftarrow\ R \cdot (\mathsf{img}\ T) \subseteq R \vee S \cdot (\mathsf{img}\ T) \subseteq S$	(5.104)
Pairing-fusion (functions)	$\langle R, S\rangle \cdot f \ = \ \langle R \cdot f, S \cdot f\rangle$	(5.105)
Pairing and converse	$\langle R, S\rangle^\circ \cdot \langle X, Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y)$	(5.108)
Kernel of pairing	$\mathsf{ker}\ \langle R, S\rangle \ = \ \mathsf{ker}\ R \cap \mathsf{ker}\ S$	(5.111)
Functor-$\times$ def	$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2\rangle$	(5.107)
Functor-$\times$-id	$id \times id = id$	(5.112)
Functor-$\times$-composition	$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q)$	(5.113)
Functor-$\times$ absorption	$(R \times S) \cdot \langle P, Q\rangle = \langle R \cdot P, S \cdot Q\rangle$	(5.106)
Functor-$\times$-division	$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k}$	(5.127)
Pairing division	$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f, h\rangle}{\langle g, k\rangle}$	(5.109)

COPRODUCTS

Universal	$X = [R, S] \quad \Leftrightarrow \quad \begin{cases} X \cdot i_1 = R \\ X \cdot i_2 = S \end{cases}$	(5.114)
Definition	$[R, S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ$	(5.117)
Reflexion	$\mathsf{img}\ i_1 \cup \mathsf{img}\ i_2 = id$	(5.115)
Disjointness	$i_1^\circ \cdot i_2 = \bot$	(5.116)
Either and converse	$[R, S] \cdot [T, U]^\circ \quad = \quad (R \cdot T^\circ) \cup (S \cdot U^\circ)$	(5.121)
Image of either	$\mathsf{img}\ [R, S] \quad = \quad \mathsf{img}\ R \cup \mathsf{img}\ S$	(5.124)
Exchange law	$[\langle R, S\rangle, \langle T, V\rangle] \quad = \quad \langle [R, T], [S, V]\rangle$	(5.122)
Functor-$+$-def	$R + S = [i_1 \cdot R, i_2 \cdot S]$	(5.119)
Functor-$+$-converse	$(R + S)^\circ = R^\circ + S^\circ$	(5.123)
Functor-$+$-division	$\frac{f}{g} + \frac{h}{k} = \frac{f + h}{g + k}$	(5.128)

INJECTIVITY PREORDER

Definition	$R \leqslant S \quad \equiv \quad \mathsf{ker}\ S \ \subseteq \ \mathsf{ker}\ R$	(5.234)
Join	$\langle R, S\rangle \leqslant X \quad \equiv \quad R \leqslant X \wedge S \leqslant X$	(5.235)

Cancellation $R \leqslant \langle R,S \rangle$ and $S \leqslant \langle R,S \rangle$. (5.236)

Shunting $R \cdot g \leqslant S \quad \equiv \quad R \leqslant S \cdot g^\circ$ (5.237)

Meet $X \leqslant [R^\circ, S^\circ]^\circ \quad \Leftrightarrow \quad X \leqslant R \wedge X \leqslant S$ (5.238)

Either-injective $id \leqslant [R,S] \Leftrightarrow id \leqslant R \wedge id \leqslant S \wedge R^\circ \cdot S = \bot$ (5.126)

THUMB RULES

A function f is a bijection iff *its converse f° is a function g* (5.38)

- converse of *injective* is *simple* (and vice-versa) (5.43)

- converse of *entire* is *surjective* (and vice-versa) (5.44)

- *smaller* than injective (simple) is injective (simple) (5.82)

- *larger* than entire (surjective) is entire (surjective) (5.83)

$\langle R, id \rangle$ is always *injective*, for whatever R (5.111a)

QUANTIFIER CALCULUS

Trading-∀ $\langle \forall\, k : R \wedge S : T \rangle = \langle \forall\, k : R : S \Rightarrow T \rangle$ (A.1)

Trading-∃ $\langle \exists\, k : R \wedge S : T \rangle = \langle \exists\, k : R : S \wedge T \rangle$ (A.2)

de Morgan $\neg\langle \forall\, k : R : T \rangle = \langle \exists\, k : R : \neg T \rangle$ (A.3)

de Morgan $\neg\langle \exists\, k : R : T \rangle = \langle \forall\, k : R : \neg T \rangle$ (A.4)

One-point-∀ $\langle \forall\, k : k = e : T \rangle = T[k := e]$ (A.5)

One-point-∃ $\langle \exists\, k : k = e : T \rangle = T[k := e]$ (A.6)

Nesting-∀ $\langle \forall\, a, b : R \wedge S : T \rangle = \langle \forall\, a : R : \langle \forall\, b : S : T \rangle \rangle$ (A.7)

Nesting-∃ $\langle \exists\, a, b : R \wedge S : T \rangle = \langle \exists\, a : R : \langle \exists\, b : S : T \rangle \rangle$ (A.8)

Rearranging-∀ $\langle \forall\, k : R \vee S : T \rangle = \langle \forall\, k : R : T \rangle \wedge \langle \forall\, k : S : T \rangle$ (A.9)

Rearranging-∀ $\langle \forall\, k : R : T \wedge S \rangle = \langle \forall\, k : R : T \rangle \wedge \langle \forall\, k : R : S \rangle$ (A.10)

Rearranging-∃ $\langle \exists\, k : R : T \vee S \rangle = \langle \exists\, k : R : T \rangle \vee \langle \exists\, k : R : S \rangle$ (A.11)

Rearranging-∃ $\langle \exists\, k : R \vee S : T \rangle = \langle \exists\, k : R : T \rangle \vee \langle \exists\, k : S : T \rangle$ (A.12)

Splitting-∀ $\langle \forall\, j : R : \langle \forall\, k : S : T \rangle \rangle = \langle \forall\, k : \langle \exists\, j : R : S \rangle : T \rangle$ (A.13)

Splitting-∃ $\langle \exists\, j : R : \langle \exists\, k : S : T \rangle \rangle = \langle \exists\, k : \langle \exists\, j : R : S \rangle : T \rangle$ (A.14)