MFES/CSI/2223 — Relation Calculus Reference Sheet

Relational composition

Pointwise def.
$$\begin{tikzcd} B & A \arrow[l, "R"'] & C \arrow[l, "S"'] \arrow[ll, bend left, "R\cdot S"] \end{tikzcd} \quad b(R \cdot S)c \;\equiv\; \langle \exists\, a \;:\; b\, R\, a :\; a\, S\, c \rangle \tag{5.11}$$

Associativity
$$R \cdot (S \cdot P) = (R \cdot S) \cdot P \tag{5.12}$$

Identity
$$\left\{ \begin{array}{l} R = R \cdot id_A \\ R = id_B \cdot R \end{array} \right. \tag{5.13}$$

Converse

Pointwise def.
$$b\, R\, a \quad \Leftrightarrow \quad a\, R^\circ\, b \tag{5.14}$$

Universal-$^\circ$
$$X^\circ \subseteq Y \quad \equiv \quad X \subseteq Y^\circ \tag{5.136}$$

Involution
$$(R^\circ)^\circ = R \tag{5.15}$$

Contravariance
$$(R \cdot S)^\circ = S^\circ \cdot R^\circ \tag{5.16}$$

Isomorphism
$$R \subseteq S \equiv R^\circ \subseteq S^\circ \tag{5.137}$$

"Guardanapo"
$$b(f^\circ \cdot R \cdot g)a \quad \equiv \quad (f\ b)R(g\ a) \tag{5.17}$$

Relation inclusion

Pointwise
$$R \subseteq S \quad \mathit{iff} \quad \langle \forall\, a, b \;::\; b\, R\, a \Rightarrow b\, S\, a \rangle \tag{5.19}$$

Reflexion
$$R \subseteq R \tag{5.21}$$

Transitivity
$$R \subseteq S \wedge S \subseteq T \quad \Rightarrow \quad R \subseteq T \tag{5.22}$$

Top and bottom
$$\bot \;\subseteq\; R \;\subseteq\; \top \tag{5.25}$$

Absorption
$$R \cdot \bot = \bot \cdot R = \bot \tag{5.26}$$

Relation equality

Pointwise
$$R = S \quad \mathit{iff} \quad \langle \forall\, a, b \;:\; a \in A \wedge b \in B :\; b\, R\, a \Leftrightarrow b\, S\, a \rangle \tag{5.18}$$

Indirect equality
$$\begin{array}{rcl} R = S & \equiv & \langle \forall\, X \;::\; (X \subseteq R \Leftrightarrow X \subseteq S) \rangle \\ & \equiv & \langle \forall\, X \;::\; (R \subseteq X \Leftrightarrow S \subseteq X) \rangle \end{array} \tag{5.24}$$

"Ping-pong"
$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R \tag{5.20}$$

Relation taxonomy

Kernel $\quad \mathsf{ker}\ R = R^{\circ} \cdot R$ (5.32)

Image $\quad \mathsf{img}\ R \overset{\mathrm{def}}{=} R \cdot R^{\circ}$ (5.33)

Duality (5.34, 5.35)

$$\begin{array}{rcl} \mathsf{ker}\ (R^{\circ}) & = & \mathsf{img}\ R \\ \mathsf{img}\ (R^{\circ}) & = & \mathsf{ker}\ R \end{array}$$

Criteria (5.36)

	Reflexive	*Coreflexive*
$\mathsf{ker}\ R$	entire R	injective R
$\mathsf{img}\ R$	surjective R	simple R

Difunctional R $\quad R \cdot \mathsf{ker}\ R \subseteq R$ (§5.22)

Functions

Shunting rules (5.46, 5.47)

$$\begin{array}{rcl} f \cdot R \subseteq S & \equiv & R \subseteq f^{\circ} \cdot S \\ R \cdot f^{\circ} \subseteq S & \equiv & R \subseteq S \cdot f \end{array}$$

Equality $\quad f \subseteq g \equiv f = g \equiv f \supseteq g$ (5.48)

Constant functions

Natural property $\quad \underline{k} \cdot R \subseteq \underline{k}$ (5.39)

Corollary $\quad \mathsf{ker}\ ! = \frac{!}{!} = \top$ (5.—)

Truth functions (5.40, 5.41)

$$\begin{array}{l} true = \underline{\mathrm{TRUE}} \\ false = \underline{\mathrm{FALSE}} \end{array}$$

Function division

Definition (5.49)

$$\frac{f}{g} = g^{\circ} \cdot f \qquad cf. \quad \begin{tikzcd} B \arrow[dr, "g"'] & & A \arrow[ll, "\frac{f}{g}"'] \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

Properties (5.50 → 5.54)

$$\begin{array}{rcl} \frac{f}{id} & = & f \\ \left(\frac{f}{g}\right)^{\circ} & = & \frac{g}{f} \\ \frac{f \cdot h}{g \cdot k} & = & k^{\circ} \cdot \frac{f}{g} \cdot h \\ \frac{f}{f} & = & \mathsf{ker}\ f \\ a \neq b & \Leftrightarrow & \frac{\underline{a}}{\underline{b}} = \bot \end{array}$$

Relation union

Pointwise definition $\quad b\ (R \cup S)\ a \equiv b\ R\ a \vee b\ S\ a$ (5.57)

Universal property $\quad R \cup S \subseteq X \equiv R \subseteq X \wedge S \subseteq X$ (5.59)

Right linearity
$$R \cdot (S \cup T) \;=\; (R \cdot S) \cup (R \cdot T) \tag{5.60}$$

Left linearity
$$(S \cup T) \cdot R \;=\; (S \cdot R) \cup (T \cdot R) \tag{5.61}$$

Converse- $\cup$
$$(R \cup S)^\circ = R^\circ \cup S^\circ \tag{5.65}$$

Top- $\cup$
$$R \cup \top = \top \tag{5.68}$$

Bottom- $\cup$
$$R \cup \bot = R \tag{5.69}$$

Union simplicity
$$M \cup N \text{ is simple} \;\equiv\; M, N \text{ are simple and } M \cdot N^\circ \subseteq id \tag{5.70}$$

Relation intersection

Pointwise definition
$$b\,(R \cap S)\,a \;\equiv\; b\,R\,a \wedge b\,S\,a \tag{5.56}$$

Universal property
$$X \subseteq R \cap S \;\equiv\; X \subseteq R \;\wedge\; X \subseteq S \tag{5.58}$$

Distribution (1)
$$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \;\Leftarrow\; \left\{ \begin{array}{c} Q \cdot \mathsf{img}\ R \subseteq Q \\ \vee \\ S \cdot \mathsf{img}\ R \subseteq S \end{array} \right. \tag{5.62}$$

Distribution (2)
$$R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \;\Leftarrow\; \left\{ \begin{array}{c} (\mathsf{ker}\ R) \cdot Q \subseteq Q \\ \vee \\ (\mathsf{ker}\ R) \cdot S \subseteq S \end{array} \right. \tag{5.63}$$

Distribution (3)
$$g^\circ \cdot (R \cap S) \cdot f \;=\; g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f \tag{5.71}$$

Converse- $\cap$
$$(R \cap S)^\circ = R^\circ \cap S^\circ \tag{5.64}$$

Bottom- $\cap$
$$R \cap \bot = \bot \tag{5.66}$$

Top- $\cap$
$$R \cap \top = R \tag{5.67}$$

Misc.
$$k^\circ \cdot (f \cup g) = \tfrac{f}{k} \cup \tfrac{g}{k} \;\;,\;\; k^\circ \cdot (f \cap g) = \tfrac{f}{k} \cap \tfrac{g}{k} \tag{5.73}$$

Relation division

Universal-/
$$Z \cdot Y \subseteq X \;\equiv\; Z \subseteq X/Y \tag{5.157}$$

Pointwise-/
$$c\,(X/Y)\,a \;\equiv\; \langle \forall\, b \;:\; a\,Y\,b :\; c\,X\,b \rangle \tag{5.158}$$
$$\begin{tikzcd} & a \arrow[dl, "X/Y"'] \\ c & b \arrow[l, "X"] \arrow[u, "Y"'] \end{tikzcd}$$

Universal-
$$X \cdot Z \subseteq Y \;\Leftrightarrow\; Z \subseteq X \setminus Y \tag{5.159}$$

Pointwise-
$$a(X \setminus Y)c \;\equiv\; \langle \forall\, b \;:\; b\,X\,a :\; b\,Y\,c \rangle \tag{5.160}$$

Misc.
$$X \cdot f \;=\; X / f^\circ \tag{5.162}$$
$$f \setminus X \;=\; f^\circ \cdot X \tag{5.163}$$

$$X / \bot \;=\; \top \tag{5.164}$$

$$X / id \;=\; X \tag{5.165}$$

$$R \setminus (f^\circ \cdot S) \;=\; f \cdot R \setminus S \tag{5.166}$$

$$R \setminus \top \cdot S \;=\; ! \cdot R \setminus ! \cdot S \tag{5.167}$$

$$R \,/\, (S \cup P) \;=\; R \,/\, S \cap R \,/\, P \tag{5.168}$$

Relation difference, implication and negation

Universal-$(-)$ $$X - R \subseteq Y \;\equiv\; X \subseteq Y \cup R \tag{5.138}$$

Pointwise-$\Rightarrow$ $$b(R \Rightarrow S)a \;\equiv\; (b\,R\,a) \Rightarrow (b\,S\,a) \tag{5.147}$$

Universal-$\Rightarrow$ $$R \cap X \subseteq Y \;\equiv\; X \subseteq (R \Rightarrow Y) \tag{5.148}$$

Distribution $$f^\circ \cdot (R \Rightarrow S) \cdot g \;=\; (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g) \tag{5.154}$$

Definition-$\neg$ $$\neg R = (R \Rightarrow \bot) \tag{5.—}$$

Pointwise-$\neg$ $$b\,(\neg R)\,a \;\Leftrightarrow\; \neg\,(b\,R\,a) \tag{5.—}$$

Complementation $$R \cup \neg R = \top \tag{5.152}$$

Difference versus implication $$\top - R \;\subseteq\; R \Rightarrow \bot \tag{5.153}$$

de Morgan $$\neg(R \cup S) = (\neg R) \cap (\neg S) \tag{5.154}$$

Schröder's rule $$\neg Q \cdot S^\circ \;\subseteq\; \neg R \;\Leftrightarrow\; R^\circ \cdot \neg Q \;\subseteq\; \neg S \tag{5.151}$$

Relation shrinking

Definition-shrink $$R \upharpoonright S \;=\; R \cap S / R^\circ \tag{5.183}$$

Universal-shrink $$X \subseteq R \upharpoonright S \;\equiv\; X \subseteq R \;\wedge\; X \cdot R^\circ \subseteq S \tag{5.182}$$

Monotonicity

Composition $$R \;\subseteq\; S \wedge T \;\subseteq\; U \;\Rightarrow\; R \cdot T \subseteq S \cdot U \tag{5.78}$$

Converse $$R \;\subseteq\; S \;\Rightarrow\; R^\circ \;\subseteq\; S^\circ \tag{5.79}$$

Intersection $$R \subseteq S \wedge U \subseteq V \;\Rightarrow\; R \cap U \subseteq S \cap V \tag{5.80}$$

Union $$R \subseteq S \wedge U \subseteq V \;\Rightarrow\; R \cup U \subseteq S \cup V \tag{5.81}$$

Endo-relation taxonomy

Reflexive $$id \subseteq R \tag{5.84}$$

Coreflexive $$R \subseteq id \tag{5.85}$$

Transitive $$R \cdot R \subseteq R \tag{5.86}$$

Symmetric $$R \subseteq R^\circ (\equiv R = R^\circ) \tag{5.87}$$

Anti-symmetric $$R \cap R^\circ \ \subseteq id \tag{5.88}$$

Irreflexive $$R \cap id = \bot \tag{5.89}$$

Connected $$R \cup R^\circ = \top \tag{5.90}$$

Pairing ("splits")

Pairing-pointwise $$(a,b)\ \langle R,S\rangle\ c \ \Leftrightarrow\ a\ R\ c \wedge b\ S\ c \tag{5.101}$$

Pairing-def $$\langle R,S\rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S \tag{5.102}$$

Universal-pairing $$X \subseteq \langle R,S\rangle \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{array} \right. \tag{5.103}$$

Pairing-fusion $$\begin{array}{l} \langle R,S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle \\ \quad \Leftarrow\ R \cdot (\mathsf{img}\ T) \subseteq R \vee S \cdot (\mathsf{img}\ T) \subseteq S \end{array} \tag{5.104}$$

Pairing-fusion (functions) $$\langle R,S\rangle \cdot f \ = \ \langle R \cdot f, S \cdot f\rangle \tag{5.105}$$

Pairing and converse $$\langle R,S\rangle^\circ \cdot \langle X,Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y) \tag{5.108}$$

Kernel of pairing $$\mathsf{ker}\ \langle R,S\rangle \ = \ \mathsf{ker}\ R \cap \mathsf{ker}\ S \tag{5.111}$$

Functor-× def $$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2\rangle \tag{5.107}$$

Functor-×-id $$id \times id = id \tag{5.112}$$

Functor-×-composition $$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q) \tag{5.113}$$

Functor-× absorption $$(R \times S) \cdot \langle P,Q\rangle = \langle R \cdot P, S \cdot Q\rangle \tag{5.106}$$

Functor-×-division $$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k} \tag{5.127}$$

Pairing division $$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f,h\rangle}{\langle g,k\rangle} \tag{5.109}$$

Coproducts

Universal $$X = [R,S] \quad \Leftrightarrow \quad \left\{ \begin{array}{l} X \cdot i_1 = R \\ X \cdot i_2 = S \end{array} \right. \tag{5.114}$$

Definition $[R,S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ$ (5.117)

Reflexion $\mathsf{img}\ i_1 \cup \mathsf{img}\ i_2 = id$ (5.115)

Disjointness $i_1^\circ \cdot i_2 = \bot$ (5.116)

Either and converse $[R,S] \cdot [T,U]^\circ \quad = \quad (R \cdot T^\circ) \cup (S \cdot U^\circ)$ (5.121)

Image of either $\mathsf{img}\ [R,S] \quad = \quad \mathsf{img}\ R \cup \mathsf{img}\ S$ (5.124)

Exchange law $[\langle R,S\rangle , \langle T,V\rangle] \quad = \quad \langle [R\ ,T], [S\ ,V]\rangle$ (5.122)

Functor-+-def $R + S = [i_1 \cdot R\ , i_2 \cdot S]$ (5.119)

Functor-+-converse $(R+S)^\circ = R^\circ + S^\circ$ (5.123)

Functor-+-division $\frac{f}{g} + \frac{h}{k} = \frac{f+h}{g+k}$ (5.128)

Injectivity preorder

Definition $R \leqslant S \quad \equiv \quad \mathsf{ker}\ S \subseteq \mathsf{ker}\ R$ (5.234)

Join $\langle R,S\rangle \leqslant X \quad \equiv \quad R \leqslant X \wedge S \leqslant X$ (5.235)

Cancellation $R \leqslant \langle R,S\rangle$ and $S \leqslant \langle R,S\rangle$. (5.236)

Shunting $R \cdot g \leqslant S \quad \equiv \quad R \leqslant S \cdot g^\circ$ (5.237)

Meet $X \leqslant [R^\circ ,S^\circ]^\circ \quad \Leftrightarrow \quad X \leqslant R \wedge X \leqslant S$ (5.238)

Either-injective $id \leqslant [R,S] \ \Leftrightarrow \ id \leqslant R \wedge id \leqslant S \wedge R^\circ \cdot S = \bot$ (5.126)

Thumb rules

A function f *is a bijection* iff *its converse* f° *is a function* g (5.38)

- converse of *injective* is *simple* (and vice-versa) (5.43)

- converse of *entire* is *surjective* (and vice-versa) (5.44)

- *smaller* than injective (simple) is injective (simple) (5.82)

- *larger* than entire (surjective) is entire (surjective) (5.83)

$\langle R, id\rangle$ is always *injective*, for whatever R (5.111a)

Quantifier calculus

Trading-$\forall$ $\langle \forall\ k\ :\ R \wedge S :\ T\rangle \quad = \quad \langle \forall\ k\ :\ R :\ S \Rightarrow T\rangle$ (A.1)

Trading-∃ $$\langle \exists\, k \;:\; R \wedge S :\; T \rangle \quad = \quad \langle \exists\, k \;:\; R :\; S \wedge T \rangle \tag{A.2}$$

de Morgan $$\neg \langle \forall\, k \;:\; R :\; T \rangle \quad = \quad \langle \exists\, k \;:\; R :\; \neg T \rangle \tag{A.3}$$

de Morgan $$\neg \langle \exists\, k \;:\; R :\; T \rangle \quad = \quad \langle \forall\, k \;:\; R :\; \neg T \rangle \tag{A.4}$$

One-point-∀ $$\langle \forall\, k \;:\; k = e :\; T \rangle \quad = \quad T[k := e] \tag{A.5}$$

One-point-∃ $$\langle \exists\, k \;:\; k = e :\; T \rangle \quad = \quad T[k := e] \tag{A.6}$$

Nesting-∀ $$\langle \forall\, a, b \;:\; R \wedge S :\; T \rangle \quad = \quad \langle \forall\, a \;:\; R :\; \langle \forall\, b \;:\; S :\; T \rangle \rangle \tag{A.7}$$

Nesting-∃ $$\langle \exists\, a, b \;:\; R \wedge S :\; T \rangle \quad = \quad \langle \exists\, a \;:\; R :\; \langle \exists\, b \;:\; S :\; T \rangle \rangle \tag{A.8}$$

Rearranging-∀ $$\langle \forall\, k \;:\; R \vee S :\; T \rangle \quad = \quad \langle \forall\, k \;:\; R :\; T \rangle \wedge \langle \forall\, k \;:\; S :\; T \rangle \tag{A.9}$$

Rearranging-∀ $$\langle \forall\, k \;:\; R :\; T \wedge S \rangle \quad = \quad \langle \forall\, k \;:\; R :\; T \rangle \wedge \langle \forall\, k \;:\; R :\; S \rangle \tag{A.10}$$

Rearranging-∃ $$\langle \exists\, k \;:\; R :\; T \vee S \rangle \quad = \quad \langle \exists\, k \;:\; R :\; T \rangle \vee \langle \exists\, k \;:\; R :\; S \rangle \tag{A.11}$$

Rearranging-∃ $$\langle \exists\, k \;:\; R \vee S :\; T \rangle \quad = \quad \langle \exists\, k \;:\; R :\; T \rangle \vee \langle \exists\, k \;:\; S :\; T \rangle \tag{A.12}$$

Splitting-∀ $$\langle \forall\, j \;:\; R :\; \langle \forall\, k \;:\; S :\; T \rangle \rangle \quad = \quad \langle \forall\, k \;:\; \langle \exists\, j \;:\; R :\; S \rangle :\; T \rangle \tag{A.13}$$

Splitting-∃ $$\langle \exists\, j \;:\; R :\; \langle \exists\, k \;:\; S :\; T \rangle \rangle \quad = \quad \langle \exists\, k \;:\; \langle \exists\, j \;:\; R :\; S \rangle :\; T \rangle \tag{A.14}$$