MFES/CSI — Relation Calculus Reference Sheet

Relational composition

Pointwise def.

$$B \xleftarrow{R} A \xleftarrow{S} C \quad (R \cdot S : C \to B) \qquad b(R \cdot S)c \;\equiv\; \langle \exists\, a \;:\; b\, R\, a :\; a\, S\, c \rangle \tag{5.11}$$

Associativity

$$R \cdot (S \cdot P) = (R \cdot S) \cdot P \tag{5.12}$$

Identity

$$\left\{ \begin{array}{l} R = R \cdot id_A \\ R = id_B \cdot R \end{array} \right. \tag{5.13}$$

Converse

Pointwise def.

$$b\, R\, a \quad \Leftrightarrow \quad a\, R^\circ\, b \tag{5.14}$$

Universal-°

$$X^\circ \subseteq Y \quad \equiv \quad X \subseteq Y^\circ \tag{5.136}$$

Involution

$$(R^\circ)^\circ = R \tag{5.15}$$

Contravariance

$$(R \cdot S)^\circ = S^\circ \cdot R^\circ \tag{5.16}$$

Isomorphism

$$R \subseteq S \equiv R^\circ \subseteq S^\circ \tag{5.137}$$

“Guardanapo”

$$b(f^\circ \cdot R \cdot g)a \quad \equiv \quad (f\ b)R(g\ a) \tag{5.17}$$

Relation inclusion

Pointwise

$$R \subseteq S \quad \mathit{iff} \quad \langle \forall\, a, b \;::\; b\, R\, a \;\Rightarrow\; b\, S\, a \rangle \tag{5.19}$$

Reflexion

$$R \subseteq R \tag{5.21}$$

Transitivity

$$R \subseteq S \wedge S \subseteq T \quad \Rightarrow \quad R \subseteq T \tag{5.22}$$

Top and bottom

$$\bot \;\subseteq\; R \;\subseteq\; \top \tag{5.25}$$

Absorption

$$R \cdot \bot = \bot \cdot R = \bot \tag{5.26}$$

Relation equality

Pointwise

$$R = S \quad \mathit{iff} \quad \langle \forall\, a, b \;:\; a \in A \wedge b \in B :\; b\, R\, a \Leftrightarrow b\, S\, a \rangle \tag{5.18}$$

Indirect equality

$$\begin{array}{rcl} R = S & \equiv & \langle \forall\, X \;::\; (X \subseteq R \Leftrightarrow X \subseteq S) \rangle \\ & \equiv & \langle \forall\, X \;::\; (R \subseteq X \Leftrightarrow S \subseteq X) \rangle \end{array} \tag{5.24}$$

“Ping-pong”

$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R \tag{5.20}$$

Relation taxonomy

Kernel $\ker R = R^\circ \cdot R$ (5.32)

Image $\mathrm{img}\ R \stackrel{\text{def}}{=} R \cdot R^\circ$ (5.33)

Duality (5.34, 5.35)

$$\ker\,(R^\circ) = \mathrm{img}\ R$$
$$\mathrm{img}\,(R^\circ) = \ker R$$

Criteria (5.36)

	Reflexive	*Coreflexive*
$\ker R$	entire R	injective R
$\mathrm{img}\ R$	surjective R	simple R

Functions

Shunting rules (5.46, 5.47)

$$f \cdot R \subseteq S \quad\equiv\quad R \subseteq f^\circ \cdot S$$
$$R \cdot f^\circ \subseteq S \quad\equiv\quad R \subseteq S \cdot f$$

Equality $f \subseteq g \;\equiv\; f = g \;\equiv\; f \supseteq g$ (5.48)

Constant functions

Natural property $\underline{k} \cdot R \;\subseteq\; \underline{k}$ (5.39)

Corollary $\ker\ ! = \frac{!}{!} = \top$ (5.—)

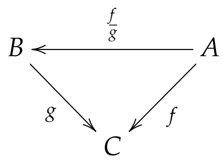
Truth functions (5.40, 5.41)

$$true = \underline{\mathrm{TRUE}}$$
$$false = \underline{\mathrm{FALSE}}$$

Function division

Definition (5.49)

$$\frac{f}{g} \;=\; g^\circ \cdot f \qquad cf.$$



Properties (5.50 → 5.54)

$$\frac{f}{id} = f$$
$$\left(\frac{f}{g}\right)^\circ = \frac{g}{f}$$
$$\frac{f \cdot h}{g \cdot k} = k^\circ \cdot \frac{f}{g} \cdot h$$
$$\frac{f}{f} = \ker f$$
$$a \neq b \;\Leftrightarrow\; \frac{\underline{a}}{\underline{b}} = \bot$$

Relation union

Pointwise definition $b\,(R \cup S)\,a \;\equiv\; b\,R\,a \vee b\,S\,a$ (5.57)

Universal property $R \cup S \;\subseteq\; X \;\equiv\; R \subseteq X \;\wedge\; S \subseteq X$ (5.59)

Right linearity $R \cdot (S \cup T) \;=\; (R \cdot S) \cup (R \cdot T)$ (5.60)

Left linearity $$(S \cup T) \cdot R \quad = \quad (S \cdot R) \cup (T \cdot R) \tag{5.61}$$

Converse-$\cup$ $$(R \cup S)^\circ = R^\circ \cup S^\circ \tag{5.65}$$

Top-$\cup$ $$R \cup \top = \top \tag{5.68}$$

Bottom-$\cup$ $$R \cup \bot = R \tag{5.69}$$

Union simplicity $$M \cup N \text{ is simple} \quad \equiv \quad M, N \text{ are simple and } M \cdot N^\circ \subseteq id \tag{5.70}$$

Relation intersection

Pointwise definition $$b\,(R \cap S)\,a \quad \equiv \quad b\,R\,a \wedge b\,S\,a \tag{5.56}$$

Universal property $$X \subseteq R \cap S \quad \equiv \quad X \subseteq R \ \wedge \ X \subseteq S \tag{5.58}$$

Distribution (1) $$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \quad \Leftarrow \quad \left\{ \begin{array}{c} Q \cdot \mathsf{img}\ R \subseteq Q \\ \vee \\ S \cdot \mathsf{img}\ R \subseteq S \end{array} \right. \tag{5.62}$$

Distribution (2) $$R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \quad \Leftarrow \quad \left\{ \begin{array}{c} (\mathsf{ker}\ R) \cdot Q \subseteq Q \\ \vee \\ (\mathsf{ker}\ R) \cdot S \subseteq S \end{array} \right. \tag{5.63}$$

Distribution (3) $$g^\circ \cdot (R \cap S) \cdot f \quad = \quad g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f \tag{5.71}$$

Converse-$\cap$ $$(R \cap S)^\circ = R^\circ \cap S^\circ \tag{5.64}$$

Bottom-$\cap$ $$R \cap \bot = \bot \tag{5.66}$$

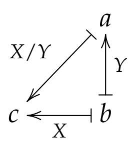
Top-$\cap$ $$R \cap \top = R \tag{5.67}$$

Misc. $$k^\circ \cdot (f \cup g) = \tfrac{f}{k} \cup \tfrac{g}{k} \quad , \quad k^\circ \cdot (f \cap g) = \tfrac{f}{k} \cap \tfrac{g}{k} \tag{5.73}$$

Relation division

Universal-$/$ $$Z \cdot Y \subseteq X \ \equiv \ Z \subseteq X/Y \tag{5.157}$$

Pointwise-$/$ $$c\,(X/Y)\,a \ \equiv \ \langle \forall\, b \ : \ a\,Y\,b : \ c\,X\,b \rangle \tag{5.158}$$



Universal-$\backslash$ $$X \cdot Z \subseteq Y \ \Leftrightarrow \ Z \subseteq X \setminus Y \tag{5.159}$$

Pointwise-$\backslash$ $$a(X \setminus Y)c \quad \equiv \quad \langle \forall\, b \ : \ b\,X\,a : \ b\,Y\,c \rangle \tag{5.160}$$

Misc. $$X \cdot f \quad = \quad X/f^\circ \tag{5.162}$$

$$f \setminus X \quad = \quad f^\circ \cdot X \tag{5.163}$$

$$X/\bot \quad = \quad \top \tag{5.164}$$

$$X / id = X \quad (5.165)$$

$$R \setminus (f^\circ \cdot S) = f \cdot R \setminus S \quad (5.166)$$

$$R \setminus \top \cdot S = ! \cdot R \setminus ! \cdot S \quad (5.167)$$

$$R / (S \cup P) = R / S \cap R / P \quad (5.168)$$

Relation difference, implication and negation

Universal-$(-)$	$X - R \subseteq Y \equiv X \subseteq Y \cup R$	(5.138)
Pointwise-$\Rightarrow$	$b(R \Rightarrow S)a \equiv (b\ R\ a) \Rightarrow (b\ S\ a)$	(5.147)
Universal-$\Rightarrow$	$R \cap X \subseteq Y \equiv X \subseteq (R \Rightarrow Y)$	(5.148)
Distribution	$f^\circ \cdot (R \Rightarrow S) \cdot g = (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g)$	(5.154)
Definition-$\neg$	$\neg R = (R \Rightarrow \bot)$	(5.—)
Pointwise-$\neg$	$b\ (\neg R)\ a \Leftrightarrow \neg\ (b\ R\ a)$	(5.—)
Complementation	$R \cup \neg R = \top$	(5.152)
Difference versus implication	$\top - R \subseteq R \Rightarrow \bot$	(5.153)
de Morgan	$\neg(R \cup S) = (\neg R) \cap (\neg S)$	(5.154)
Schröder's rule	$\neg Q \cdot S^\circ \subseteq \neg R \Leftrightarrow R^\circ \cdot \neg Q \subseteq \neg S$	(5.151)

Monotonicity

Composition	$R \subseteq S \wedge T \subseteq U \Rightarrow R \cdot T \subseteq S \cdot U$	(5.78)
Converse	$R \subseteq S \Rightarrow R^\circ \subseteq S^\circ$	(5.79)
Intersection	$R \subseteq S \wedge U \subseteq V \Rightarrow R \cap U \subseteq S \cap V$	(5.80)
Union	$R \subseteq S \wedge U \subseteq V \Rightarrow R \cup U \subseteq S \cup V$	(5.81)

Endo-relation taxonomy

Reflexive	$id \subseteq R$	(5.84)
Coreflexive	$R \subseteq id$	(5.85)
Transitive	$R \cdot R \subseteq R$	(5.86)
Symmetric	$R \subseteq R^\circ (\equiv R = R^\circ)$	(5.87)

Anti-symmetric	$R \cap R^\circ \subseteq id$	(5.88)
Irreflexive	$R \cap id = \bot$	(5.89)
Connected	$R \cup R^\circ = \top$	(5.90)

Pairing ("splits")

Pairing-pointwise	$(a, b)\ \langle R, S\rangle\ c \ \Leftrightarrow\ a\ R\ c \wedge b\ S\ c$	(5.101)
Pairing-def	$\langle R, S\rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S$	(5.102)
Universal-pairing	$X \subseteq \langle R, S\rangle \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{array} \right.$	(5.103)
Pairing-fusion	$\langle R, S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle \Leftarrow R \cdot (\mathsf{img}\ T) \subseteq R \vee S \cdot (\mathsf{img}\ T) \subseteq S$	(5.104)
Pairing-fusion (functions)	$\langle R, S\rangle \cdot f \ = \ \langle R \cdot f, S \cdot f\rangle$	(5.105)
Pairing and converse	$\langle R, S\rangle^\circ \cdot \langle X, Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y)$	(5.108)
Kernel of pairing	$\mathsf{ker}\ \langle R, S\rangle \ = \ \mathsf{ker}\ R \cap \mathsf{ker}\ S$	(5.111)
Functor-× def	$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2\rangle$	(5.107)
Functor-×-id	$id \times id = id$	(5.112)
Functor-×-composition	$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q)$	(5.113)
Functor-× absorption	$(R \times S) \cdot \langle P, Q\rangle = \langle R \cdot P, S \cdot Q\rangle$	(5.106)
Functor-×-division	$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k}$	(5.127)
Pairing division	$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f, h\rangle}{\langle g, k\rangle}$	(5.109)

Coproducts

Universal	$X = [R, S] \quad \Leftrightarrow \quad \left\{ \begin{array}{l} X \cdot i_1 = R \\ X \cdot i_2 = S \end{array} \right.$	(5.114)
Definition	$[R, S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ$	(5.117)
Reflexion	$\mathsf{img}\ i_1 \cup \mathsf{img}\ i_2 = id$	(5.115)
Disjointness	$i_1^\circ \cdot i_2 = \bot$	(5.116)
Either and converse	$[R, S] \cdot [T, U]^\circ \quad = \quad (R \cdot T^\circ) \cup (S \cdot U^\circ)$	(5.121)
Image of either	$\mathsf{img}\ [R, S] \quad = \quad \mathsf{img}\ R \cup \mathsf{img}\ S$	(5.124)
Exchange law	$[\langle R, S\rangle, \langle T, V\rangle] \quad = \quad \langle [R, T], [S, V]\rangle$	(5.122)

Functor-+-def $R + S = [i_1 \cdot R \,, i_2 \cdot S]$ (5.119)

Functor-+-converse $(R + S)^\circ = R^\circ + S^\circ$ (5.123)

Functor-+-division $\frac{f}{g} + \frac{h}{k} = \frac{f+h}{g+k}$ (5.128)

INJECTIVITY PREORDER

Definition $R \leqslant S \quad \equiv \quad \mathsf{ker}\, S \subseteq \mathsf{ker}\, R$ (5.234)

Join $\langle R, S \rangle \leqslant X \quad \equiv \quad R \leqslant X \wedge S \leqslant X$ (5.235)

Cancellation $R \leqslant \langle R, S \rangle$ and $S \leqslant \langle R, S \rangle$. (5.236)

Shunting $R \cdot g \leqslant S \quad \equiv \quad R \leqslant S \cdot g^\circ$ (5.237)

Meet $X \leqslant [R^\circ \,, S^\circ]^\circ \quad \Leftrightarrow \quad X \leqslant R \wedge X \leqslant S$ (5.238)

Either-injective $id \leqslant [R \,, S] \;\Leftrightarrow\; id \leqslant R \wedge id \leqslant S \wedge R^\circ \cdot S = \bot$ (5.126)

THUMB RULES

A function f is a bijection iff *its converse f° is a function g* (5.38)

- converse of *injective* is *simple* (and vice-versa) (5.43)

- converse of *entire* is *surjective* (and vice-versa) (5.44)

- *smaller* than injective (simple) is injective (simple) (5.82)

- *larger* than entire (surjective) is entire (surjective) (5.83)

$\langle R, id \rangle$ is always *injective*, for whatever R (5.111a)

QUANTIFIER CALCULUS

Trading-∀ $\langle \forall\, k \;:\; R \wedge S :\; T \rangle \quad = \quad \langle \forall\, k \;:\; R :\; S \Rightarrow T \rangle$ (A.1)

Trading-∃ $\langle \exists\, k \;:\; R \wedge S :\; T \rangle \quad = \quad \langle \exists\, k \;:\; R :\; S \wedge T \rangle$ (A.2)

de Morgan $\neg \langle \forall\, k \;:\; R :\; T \rangle \quad = \quad \langle \exists\, k \;:\; R :\; \neg T \rangle$ (A.3)

de Morgan $\neg \langle \exists\, k \;:\; R :\; T \rangle \quad = \quad \langle \forall\, k \;:\; R :\; \neg T \rangle$ (A.4)

One-point-∀ $\langle \forall\, k \;:\; k = e :\; T \rangle \quad = \quad T[k := e]$ (A.5)

One-point-∃ $\langle \exists\, k \;:\; k = e :\; T \rangle \quad = \quad T[k := e]$ (A.6)

Nesting-∀ $\langle \forall\, a, b \;:\; R \wedge S :\; T \rangle \quad = \quad \langle \forall\, a \;:\; R :\; \langle \forall\, b \;:\; S :\; T \rangle \rangle$ (A.7)

Nesting-∃ $$\langle \exists\, a, b \,:\, R \wedge S :\, T \rangle \;=\; \langle \exists\, a \,:\, R :\, \langle \exists\, b \,:\, S :\, T \rangle \rangle \tag{A.8}$$

Rearranging-∀ $$\langle \forall\, k \,:\, R \vee S :\, T \rangle \;=\; \langle \forall\, k \,:\, R :\, T \rangle \wedge \langle \forall\, k \,:\, S :\, T \rangle \tag{A.9}$$

Rearranging-∀ $$\langle \forall\, k \,:\, R :\, T \wedge S \rangle \;=\; \langle \forall\, k \,:\, R :\, T \rangle \wedge \langle \forall\, k \,:\, R :\, S \rangle \tag{A.10}$$

Rearranging-∃ $$\langle \exists\, k \,:\, R :\, T \vee S \rangle \;=\; \langle \exists\, k \,:\, R :\, T \rangle \vee \langle \exists\, k \,:\, R :\, S \rangle \tag{A.11}$$

Rearranging-∃ $$\langle \exists\, k \,:\, R \vee S :\, T \rangle \;=\; \langle \exists\, k \,:\, R :\, T \rangle \vee \langle \exists\, k \,:\, S :\, T \rangle \tag{A.12}$$

Splitting-∀ $$\langle \forall\, j \,:\, R :\, \langle \forall\, k \,:\, S :\, T \rangle \rangle \;=\; \langle \forall\, k \,:\, \langle \exists\, j \,:\, R :\, S \rangle :\, T \rangle \tag{A.13}$$

Splitting-∃ $$\langle \exists\, j \,:\, R :\, \langle \exists\, k \,:\, S :\, T \rangle \rangle \;=\; \langle \exists\, k \,:\, \langle \exists\, j \,:\, R :\, S \rangle :\, T \rangle \tag{A.14}$$