MFES/CSI — Relation Calculus Reference Sheet

Relational composition

Pointwise def.

$$\begin{tikzcd} B & A \arrow[l, "R"'] & C \arrow[l, "S"'] \arrow[ll, bend left, "R \cdot S"] \end{tikzcd} \quad b(R \cdot S)c \equiv \langle \exists\, a : b\, R\, a : a\, S\, c \rangle \tag{5.11}$$

Associativity

$$R \cdot (S \cdot P) = (R \cdot S) \cdot P \tag{5.12}$$

Identity

$$\left\{ \begin{array}{l} R = R \cdot id_A \\ R = id_B \cdot R \end{array} \right. \tag{5.13}$$

Converse

Pointwise def.

$$b\, R\, a \quad \Leftrightarrow \quad a\, R^\circ\, b \tag{5.14}$$

Universal-$\circ$

$$X^\circ \subseteq Y \quad \equiv \quad X \subseteq Y^\circ \tag{5.136}$$

Involution

$$(R^\circ)^\circ = R \tag{5.15}$$

Contravariance

$$(R \cdot S)^\circ = S^\circ \cdot R^\circ \tag{5.16}$$

Isomorphism

$$R \subseteq S \equiv R^\circ \subseteq S^\circ \tag{5.137}$$

“Guardanapo”

$$b(f^\circ \cdot R \cdot g)a \quad \equiv \quad (f\ b)R(g\ a) \tag{5.17}$$

Relation inclusion

Pointwise

$$R \subseteq S \quad \textit{iff} \quad \langle \forall\, a, b :: b\, R\, a \Rightarrow b\, S\, a \rangle \tag{5.19}$$

Reflexion

$$R \subseteq R \tag{5.21}$$

Transitivity

$$R \subseteq S \wedge S \subseteq T \quad \Rightarrow \quad R \subseteq T \tag{5.22}$$

Top and bottom

$$\bot \subseteq R \subseteq \top \tag{5.25}$$

Absorption

$$R \cdot \bot = \bot \cdot R = \bot \tag{5.26}$$

Relation equality

Pointwise

$$R = S \quad \textit{iff} \quad \langle \forall\, a, b : a \in A \wedge b \in B : b\, R\, a \Leftrightarrow b\, S\, a \rangle \tag{5.18}$$

Indirect equality

$$\begin{array}{rcl} R = S & \equiv & \langle \forall\, X :: (X \subseteq R \Leftrightarrow X \subseteq S) \rangle \\ & \equiv & \langle \forall\, X :: (R \subseteq X \Leftrightarrow S \subseteq X) \rangle \end{array} \tag{5.24}$$

“Ping-pong”

$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R \tag{5.20}$$

Relation taxonomy

Kernel

$$\mathsf{ker}\ R = R^{\circ} \cdot R \qquad (5.32)$$

Image

$$\mathsf{img}\ R \stackrel{\mathrm{def}}{=} R \cdot R^{\circ} \qquad (5.33)$$

Duality

$$\begin{array}{rcl} \mathsf{ker}\ (R^{\circ}) & = & \mathsf{img}\ R \\ \mathsf{img}\ (R^{\circ}) & = & \mathsf{ker}\ R \end{array} \qquad (5.34, 5.35)$$

Criteria (5.36)

	Reflexive	*Coreflexive*
ker R	entire R	injective R
img R	surjective R	simple R

Functions

Shunting rules

$$\begin{array}{rcl} f \cdot R \subseteq S & \equiv & R \subseteq f^{\circ} \cdot S \\ R \cdot f^{\circ} \subseteq S & \equiv & R \subseteq S \cdot f \end{array} \qquad (5.46, 5.47)$$

Equality

$$f \subseteq g \ \equiv\ f = g \ \equiv\ f \supseteq g \qquad (5.48)$$

Constant functions

Natural property

$$\underline{k} \cdot R \ \subseteq\ \underline{k} \qquad (5.39)$$

Corollary

$$\mathsf{ker}\ ! = \frac{!}{!} = \top \qquad (5.—)$$

Truth functions

$$\begin{array}{c} \mathit{true} = \underline{\mathsf{True}} \\ \mathit{false} = \underline{\mathsf{False}} \end{array} \qquad (5.40, 5.41)$$

Function division

Definition

$$\frac{f}{g} \ = \ g^{\circ} \cdot f \qquad cf. \qquad (5.49)$$

$$\begin{tikzcd} B \arrow[dr, "g"'] & & A \arrow[ll, "\frac{f}{g}"'] \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

Properties

$$\begin{array}{rcl} \frac{f}{id} & = & f \\ \left(\frac{f}{g}\right)^{\circ} & = & \frac{g}{f} \\ \frac{f \cdot h}{g \cdot k} & = & k^{\circ} \cdot \frac{f}{g} \cdot h \\ \frac{f}{f} & = & \mathsf{ker}\, f \\ a \neq b & \Leftrightarrow & \frac{\underline{a}}{\underline{b}} = \bot \end{array} \qquad (5.50 \rightarrow 5.54)$$

Relation union

Pointwise definition

$$b\ (R \cup S)\ a \ \equiv\ b\ R\ a \vee b\ S\ a \qquad (5.57)$$

Universal property

$$R \cup S \ \subseteq\ X \ \equiv\ R \subseteq X \ \wedge\ S \subseteq X \qquad (5.59)$$

Right linearity

$$R \cdot (S \cup T) \ = \ (R \cdot S) \cup (R \cdot T) \qquad (5.60)$$

Left linearity	$(S \cup T) \cdot R \;=\; (S \cdot R) \cup (T \cdot R)$	(5.61)
Converse- $\cup$	$(R \cup S)^\circ = R^\circ \cup S^\circ$	(5.65)
Top- $\cup$	$R \cup \top = \top$	(5.68)
Bottom- $\cup$	$R \cup \bot = R$	(5.69)
Union simplicity	$M \cup N$ is simple $\;\equiv\;$ M, N are simple and $M \cdot N^\circ \subseteq id$	(5.70)

Relation intersection

Pointwise definition	$b\ (R \cap S)\ a \;\equiv\; b\ R\ a \wedge b\ S\ a$	(5.56)
Universal property	$X \subseteq R \cap S \;\equiv\; X \subseteq R \,\wedge\, X \subseteq S$	(5.58)
Distribution (1)	$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \;\Leftarrow\; \left\{ \begin{array}{c} Q \cdot \mathsf{img}\ R \subseteq Q \\ \vee \\ S \cdot \mathsf{img}\ R \subseteq S \end{array} \right.$	(5.62)
Distribution (2)	$R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \;\Leftarrow\; \left\{ \begin{array}{c} (\mathsf{ker}\ R) \cdot Q \subseteq Q \\ \vee \\ (\mathsf{ker}\ R) \cdot S \subseteq S \end{array} \right.$	(5.63)
Distribution (3)	$g^\circ \cdot (R \cap S) \cdot f \;=\; g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f$	(5.71)
Converse- $\cap$	$(R \cap S)^\circ = R^\circ \cap S^\circ$	(5.64)
Bottom- $\cap$	$R \cap \bot = \bot$	(5.66)
Top- $\cap$	$R \cap \top = R$	(5.67)
Misc.	$k^\circ \cdot (f \cup g) = \frac{f}{k} \cup \frac{g}{k} \;\;,\;\; k^\circ \cdot (f \cap g) = \frac{f}{k} \cap \frac{g}{k}$	(5.73)

Relation division

Universal-$/$	$Z \cdot Y \subseteq X \;\equiv\; Z \subseteq X/Y$	(5.157)
Pointwise-$/$	$c\ (X/Y)\ a \;\equiv\; \langle \forall\ b \;:\; a\ Y\ b :\; c\ X\ b \rangle$ $\begin{tikzcd} & a \arrow[dl, "X/Y"'] \\ c & b \arrow[l, "X"] \arrow[u, "Y"'] \end{tikzcd}$	(5.158)
Universal-$\backslash$	$X \cdot Z \subseteq Y \;\Leftrightarrow\; Z \subseteq X \setminus Y$	(5.159)
Pointwise-$\backslash$	$a(X \setminus Y)c \;\equiv\; \langle \forall\ b \;:\; b\ X\ a :\; b\ Y\ c \rangle$	(5.160)
Misc.	$X \cdot f \;=\; X/f^\circ$	(5.162)
	$f \setminus X \;=\; f^\circ \cdot X$	(5.163)
	$X/\bot \;=\; \top$	(5.164)

$$X / id = X \tag{5.165}$$

$$R \setminus (f^\circ \cdot S) = f \cdot R \setminus S \tag{5.166}$$

$$R \setminus \top \cdot S = \,! \cdot R \setminus \,! \cdot S \tag{5.167}$$

$$R / (S \cup P) = R / S \cap R / P \tag{5.168}$$

Relation difference, implication and negation

Universal-$(-)$ $$X - R \subseteq Y \equiv X \subseteq Y \cup R \tag{5.138}$$

Pointwise-$\Rightarrow$ $$b(R \Rightarrow S)a \equiv (b\ R\ a) \Rightarrow (b\ S\ a) \tag{5.147}$$

Universal-$\Rightarrow$ $$R \cap X \subseteq Y \equiv X \subseteq (R \Rightarrow Y) \tag{5.148}$$

Distribution $$f^\circ \cdot (R \Rightarrow S) \cdot g = (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g) \tag{5.154}$$

Definition-$\neg$ $$\neg R = (R \Rightarrow \bot) \tag{5.—}$$

Pointwise-$\neg$ $$b\ (\neg R)\ a \Leftrightarrow \neg\ (b\ R\ a) \tag{5.—}$$

Complementation $$R \cup \neg R = \top \tag{5.152}$$

Difference versus implication $$\top - R \subseteq R \Rightarrow \bot \tag{5.153}$$

de Morgan $$\neg(R \cup S) = (\neg R) \cap (\neg S) \tag{5.154}$$

Schröder's rule $$\neg Q \cdot S^\circ \subseteq \neg R \Leftrightarrow R^\circ \cdot \neg Q \subseteq \neg S \tag{5.151}$$

Monotonicity

Composition $$R \subseteq S \wedge T \subseteq U \Rightarrow R \cdot T \subseteq S \cdot U \tag{5.78}$$

Converse $$R \subseteq S \Rightarrow R^\circ \subseteq S^\circ \tag{5.79}$$

Intersection $$R \subseteq S \wedge U \subseteq V \Rightarrow R \cap U \subseteq S \cap V \tag{5.80}$$

Union $$R \subseteq S \wedge U \subseteq V \Rightarrow R \cup U \subseteq S \cup V \tag{5.81}$$

Endo-relation taxonomy

Reflexive $$id \subseteq R \tag{5.84}$$

Coreflexive $$R \subseteq id \tag{5.85}$$

Transitive $$R \cdot R \subseteq R \tag{5.86}$$

Symmetric $$R \subseteq R^\circ (\equiv R = R^\circ) \tag{5.87}$$

Anti-symmetric	$R \cap R^\circ \subseteq id$	(5.88)
Irreflexive	$R \cap id = \bot$	(5.89)
Connected	$R \cup R^\circ = \top$	(5.90)

Pairing ("splits")

Pairing-pointwise	$(a,b)\ \langle R,S\rangle\ c \ \Leftrightarrow\ a\ R\ c \wedge b\ S\ c$	(5.101)
Pairing-def	$\langle R,S\rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S$	(5.102)
Universal-pairing	$X \subseteq \langle R,S\rangle \quad \Leftrightarrow \quad \begin{cases} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{cases}$	(5.103)
Pairing-fusion	$\langle R,S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle \Leftarrow R \cdot (\mathsf{img}\ T) \subseteq R \vee S \cdot (\mathsf{img}\ T) \subseteq S$	(5.104)
Pairing-fusion (functions)	$\langle R,S\rangle \cdot f \ = \ \langle R \cdot f, S \cdot f\rangle$	(5.105)
Pairing and converse	$\langle R,S\rangle^\circ \cdot \langle X,Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y)$	(5.108)
Kernel of pairing	$\mathsf{ker}\ \langle R,S\rangle \ = \ \mathsf{ker}\ R \cap \mathsf{ker}\ S$	(5.111)
Functor-$\times$ def	$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2\rangle$	(5.107)
Functor-$\times$-id	$id \times id = id$	(5.112)
Functor-$\times$-composition	$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q)$	(5.113)
Functor-$\times$ absorption	$(R \times S) \cdot \langle P,Q\rangle = \langle R \cdot P, S \cdot Q\rangle$	(5.106)
Functor-$\times$-division	$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k}$	(5.127)
Pairing division	$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f,h\rangle}{\langle g,k\rangle}$	(5.109)

Coproducts

Universal	$X = [R\,,S] \quad \Leftrightarrow \quad \begin{cases} X \cdot i_1 = R \\ X \cdot i_2 = S \end{cases}$	(5.114)
Definition	$[R\,,S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ$	(5.117)
Reflexion	$\mathsf{img}\ i_1 \cup \mathsf{img}\ i_2 = id$	(5.115)
Disjointness	$i_1^\circ \cdot i_2 = \bot$	(5.116)
Either and converse	$[R\,,S] \cdot [T\,,U]^\circ \quad = \quad (R \cdot T^\circ) \cup (S \cdot U^\circ)$	(5.121)
Image of either	$\mathsf{img}\ [R\,,S] \quad = \quad \mathsf{img}\ R \cup \mathsf{img}\ S$	(5.124)
Exchange law	$[\langle R,S\rangle\ , \langle T,V\rangle] \quad = \quad \langle [R\ ,T], [S\ ,V]\rangle$	(5.122)

Functor-+-def	$R + S = [i_1 \cdot R \,, i_2 \cdot S]$	(5.119)
Functor-+-converse	$(R+S)^\circ = R^\circ + S^\circ$	(5.123)
Functor-+-division	$\frac{f}{g} + \frac{h}{k} = \frac{f+h}{g+k}$	(5.128)

Injectivity preorder

Definition	$R \leqslant S \quad \equiv \quad \ker S \subseteq \ker R$	(5.234)
Join	$\langle R, S\rangle \leqslant X \quad \equiv \quad R \leqslant X \wedge S \leqslant X$	(5.235)
Cancellation	$R \leqslant \langle R, S\rangle$ and $S \leqslant \langle R, S\rangle$.	(5.236)
Shunting	$R \cdot g \leqslant S \quad \equiv \quad R \leqslant S \cdot g^\circ$	(5.237)
Meet	$X \leqslant [R^\circ \,, S^\circ]^\circ \quad \Leftrightarrow \quad X \leqslant R \wedge X \leqslant S$	(5.238)
Either-injective	$id \leqslant [R\,, S] \;\Leftrightarrow\; id \leqslant R \wedge id \leqslant S \wedge R^\circ \cdot S = \bot$	(5.126)

Thumb rules

A function f is a bijection iff *its converse f° is a function g*	(5.38)
- converse of *injective* is *simple* (and vice-versa)	(5.43)
- converse of *entire* is *surjective* (and vice-versa)	(5.44)
- *smaller* than injective (simple) is injective (simple)	(5.82)
- *larger* than entire (surjective) is entire (surjective)	(5.83)
$\langle R, id\rangle$ is always *injective*, for whatever R	(5.111a)

Quantifier calculus

Trading-∀	$\langle \forall\, k \;:\; R \wedge S :\; T\rangle \quad = \quad \langle \forall\, k \;:\; R :\; S \Rightarrow T\rangle$	(A.1)
Trading-∃	$\langle \exists\, k \;:\; R \wedge S :\; T\rangle \quad = \quad \langle \exists\, k \;:\; R :\; S \wedge T\rangle$	(A.2)
de Morgan	$\neg\langle \forall\, k \;:\; R :\; T\rangle \quad = \quad \langle \exists\, k \;:\; R :\; \neg T\rangle$	(A.3)
de Morgan	$\neg\langle \exists\, k \;:\; R :\; T\rangle \quad = \quad \langle \forall\, k \;:\; R :\; \neg T\rangle$	(A.4)
One-point-∀	$\langle \forall\, k \;:\; k = e :\; T\rangle \quad = \quad T[k := e]$	(A.5)
One-point-∃	$\langle \exists\, k \;:\; k = e :\; T\rangle \quad = \quad T[k := e]$	(A.6)
Nesting-∀	$\langle \forall\, a, b \;:\; R \wedge S :\; T\rangle \quad = \quad \langle \forall\, a \;:\; R :\; \langle \forall\, b \;:\; S :\; T\rangle\rangle$	(A.7)

Nesting-∃ $$\langle \exists\, a, b \;:\; R \wedge S \,:\; T \rangle \;=\; \langle \exists\, a \;:\; R \,:\; \langle \exists\, b \;:\; S \,:\; T \rangle \rangle \tag{A.8}$$

Rearranging-∀ $$\langle \forall\, k \;:\; R \vee S \,:\; T \rangle \;=\; \langle \forall\, k \;:\; R \,:\; T \rangle \wedge \langle \forall\, k \;:\; S \,:\; T \rangle \tag{A.9}$$

Rearranging-∀ $$\langle \forall\, k \;:\; R \,:\; T \wedge S \rangle \;=\; \langle \forall\, k \;:\; R \,:\; T \rangle \wedge \langle \forall\, k \;:\; R \,:\; S \rangle \tag{A.10}$$

Rearranging-∃ $$\langle \exists\, k \;:\; R \,:\; T \vee S \rangle \;=\; \langle \exists\, k \;:\; R \,:\; T \rangle \vee \langle \exists\, k \;:\; R \,:\; S \rangle \tag{A.11}$$

Rearranging-∃ $$\langle \exists\, k \;:\; R \vee S \,:\; T \rangle \;=\; \langle \exists\, k \;:\; R \,:\; T \rangle \vee \langle \exists\, k \;:\; S \,:\; T \rangle \tag{A.12}$$

Splitting-∀ $$\langle \forall\, j \;:\; R \,:\; \langle \forall\, k \;:\; S \,:\; T \rangle \rangle \;=\; \langle \forall\, k \;:\; \langle \exists\, j \;:\; R \,:\; S \rangle \,:\; T \rangle \tag{A.13}$$

Splitting-∃ $$\langle \exists\, j \;:\; R \,:\; \langle \exists\, k \;:\; S \,:\; T \rangle \rangle \;=\; \langle \exists\, k \;:\; \langle \exists\, j \;:\; R \,:\; S \rangle \,:\; T \rangle \tag{A.14}$$

BIBLIOGRAPHY