Propositional Logic

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(based on slides by Alcino Cunha)

Overview

Formal logic

- A **formal language** in which (well-formed) sentences are expressed
- A **semantics** that defines the meaning of the language expressions
- A **proof system** that provides rules for deriving valid judgements

Propositional logic

- Propositional Logic (PL) deals with **propositions** and their relationships
- **Propositions** can be either true or false
- Atomic formulas are **propositional variables** that represent propositions
- Compound formulas are built with **logical connectives**
- Each formula is either true or false for a given variable interpretation

Syntax

- Propositional variables
  - $A, B, C, \ldots$
- Logical connectives
  - $\top$ (true), $\bot$ (false), $\neg$(not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (equivalent)
- Auxiliary symbols
  - Parenthesis (,)
- Well-formed formulas follow the syntactic rules of propositional logic

Syntax

$$A, B, C, \ldots \in \mathscr{V}$$

$$\phi, \psi, \ldots \in \mathbf{Form}_{\mathscr{V}}$$

$$\begin{aligned}\phi, \psi \doteq\ & A, B, C, \ldots \\ & \mid \top \\ & \mid \bot \\ & \mid (\neg \phi) \\ & \mid (\phi \wedge \psi) \\ & \mid (\phi \vee \psi) \\ & \mid (\phi \to \psi) \\ & \mid (\phi \leftrightarrow \psi)\end{aligned}$$

- If parenthesis are omitted, connectives have precedence $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, and are right-associative

Semantics

- A truth value **assignment** to variables $\mathcal{A} : \mathcal{V} \to \{0,1\}$
- Can be extended to $\mathcal{A} : \mathbf{Form}_{\mathcal{V}} \to \{0,1\}$ to compute the truth value of a formula ϕ
- If $\mathcal{A}(\phi) = 1$ we say that ϕ **holds** under $\mathcal{A}$, denoted by $\mathcal{A} \vDash \phi$
- If $\mathcal{A}(\phi) = 0$ we say that ϕ **does not hold** under $\mathcal{A}$, denoted by $\mathcal{A} \nvDash \phi$

Truth table semantics

$\mathscr{A}(\top)$	$\mathscr{A}(\bot)$
1	0

$\mathscr{A}(\phi)$	$\mathscr{A}(\neg\phi)$
0	1
1	0

$\mathscr{A}(\phi)$	$\mathscr{A}(\psi)$	$\mathscr{A}(\phi \wedge \psi)$	$\mathscr{A}(\phi \vee \psi)$	$\mathscr{A}(\phi \to \psi)$	$\mathscr{A}(\phi \leftrightarrow \psi)$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Inductive semantics

$$\mathscr{A} \models \top$$

$$\mathscr{A} \not\models \bot$$

$$\mathscr{A} \models p \quad \text{iff} \quad \mathscr{A}(p) = 1$$

$$\mathscr{A} \models \neg\phi \quad \text{iff} \quad \mathscr{A} \not\models \phi$$

$$\mathscr{A} \models \phi \wedge \psi \quad \text{iff} \quad \mathscr{A} \models \phi \text{ and } \mathscr{A} \models \psi$$

$$\mathscr{A} \models \phi \vee \psi \quad \text{iff} \quad \mathscr{A} \models \phi \text{ or } \mathscr{A} \models \psi$$

$$\mathscr{A} \models \phi \rightarrow \psi \quad \text{iff} \quad \mathscr{A} \not\models \phi \text{ or } \mathscr{A} \models \psi$$

$$\mathscr{A} \models \phi \leftrightarrow \psi \quad \text{iff} \quad \mathscr{A} \models \phi \text{ iff } \mathscr{A} \models \psi$$

Terminology

- A formula ϕ is
  - **valid** or a **tautology** iff it holds under *all* assignments, and we write $\models \phi$
  - **satisfiable** iff it holds under *some* assignments
  - **unsatisfiable** or a **contradiction** iff it does *not* hold under *all* assignments
  - **refutable** iff it does *not* hold under *some* assignment
- A formula ϕ is valid iff $\neg\phi$ is unsatisfiable

Examples

- $B \vee (A \rightarrow \neg B)$ is valid

A	B	$\neg B$	$A \rightarrow \neg B$	$B \vee (A \rightarrow \neg B)$
0	0	1	1	1
0	1	0	1	1
1	0	1	1	1
1	1	0	0	1

- $A \rightarrow (\neg A \vee B)$ is both satisfiable and refutable

A	B	$\neg A$	$\neg A \vee B$	$A \rightarrow (\neg A \vee B)$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Consequence and equivalence

- ϕ is a consequence of ψ, denoted by $\psi \models \phi$, if for every $\mathscr{A}$ that $\mathscr{A} \models \psi$, $\mathscr{A} \models \phi$ also holds
- ϕ and ψ are equivalent, denoted by $\psi \equiv \phi$, if $\phi \models \psi$ and $\psi \models \phi$
- $\phi \models \psi$ iff $\models \phi \rightarrow \psi$ and $\phi \equiv \psi$ iff $\models \phi \leftrightarrow \psi$

Basic equivalences

$$\phi \vee \phi \equiv \phi$$
$$\phi \vee \neg\phi \equiv \top$$
$$\phi \vee \top \equiv \top$$
$$\phi \vee \bot \equiv \phi$$
$$\phi \vee \psi \equiv \psi \vee \phi$$
$$\phi \vee (\phi \wedge \delta) \equiv (\phi \vee \phi) \wedge (\phi \vee \delta)$$
$$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$$
$$\phi \wedge (\phi \vee \psi) \equiv \phi$$
$$\phi \to \psi \equiv \neg\phi \vee \psi$$

$$\phi \wedge \phi \equiv \phi$$
$$\phi \wedge \neg\phi \equiv \bot$$
$$\phi \wedge \top \equiv \phi$$
$$\phi \wedge \bot \equiv \bot$$
$$\phi \wedge \phi \equiv \psi \wedge \psi$$
$$\phi \wedge (\phi \vee \delta) \equiv (\phi \wedge \phi) \vee (\phi \wedge \delta)$$
$$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$$
$$\neg\neg\phi \equiv \phi$$
$$\phi \leftrightarrow \psi \equiv (\phi \to \psi) \wedge (\psi \to \phi)$$

Decidability

- A decision problem is **decidable** if there exists a mechanical method (aka an algorithm) for determining if it is either true or false

- The decision problem “Is ϕ satisfiable?”, also known as the **Boolean satisfiability problem** or **SAT**, is decidable

  - Naïve approach, exponential in the number of variables: enumerate all possible assignments and build the truth table for ϕ

  - Later we will see that modern **SAT solvers** implement more sophisticated methods...

- Hence the decision problem “Is ϕ valid?” is also decidable

Modelling with PL

Formal modelling

- Logic allows us to reason formally during various software development tasks:
  - Program analysis
  - Artificial inteligence
  - Hardware verification
  - Programming languages
- (The relationship is symbiotic: software provides implementations of logics)
- The system under analysis must be represented in the selected formal system
  - We call this task **formal modelling**

Allocation problems

Allocation problem

- Decide if a set of “items” can be placed in a set of “containers”
- Subject to a set of generic (often implicit) constraints
  - At least / at most one item per container
  - At least / at most one container per item
- And a set of problem specific constraints
  - A specific item does not want to be placed in a specific container
  - Two specific items do not want to be placed together
  - ...

Allocating in PL

- Declare a matrix of $|\text{Items}| \times |\text{Containers}|$ of propositional variables
  - Variable $p_{x,a}$ is true iff item x is allocated to container a
- Specify the allocation constraints with propositional formulas
- SAT solve to get allocation

Generic constraints

- Consider 3 containers $(1,2,3)$ and 2 items (a, b)
- At least one container per item

$$(p_{a,1} \vee p_{a,2} \vee p_{a,3}) \wedge (p_{b,1} \vee p_{b,2} \vee p_{b,3})$$

- At most one container per item

		Container		
		1	2	3
Item	*a*	$p_{a,1}$	$p_{a,2}$	$p_{a,3}$
	b	$p_{b,1}$	$p_{b,2}$	$p_{b,3}$

Generic constraints

- Consider 3 containers $(1,2,3)$ and *k* items ($i_0, \ldots, i_k$)
- At least one container per item

$$(p_{i_0,1} \vee p_{i_0,2} \vee p_{i_0,3}) \wedge \ldots \wedge (p_{i_k,1} \vee p_{i_k,2} \vee p_{i_k,3})$$

- At most one container per item

		Container		
		1	2	3
Item	i_0	$p_{i_0,1}$	$p_{i_0,2}$	$p_{i_0,3}$
	i_1	$p_{i_1,1}$	$p_{i_1,2}$	$p_{i_1,3}$
	…	…	…	…
	i_k	$p_{i_k,1}$	$p_{i_k,2}$	$p_{i_k,3}$

Generic constraints

- Consider *l* containers ($c_0, \ldots, c_l$) and *k* items ($a_0, \ldots, a_k$)
- At least one container per item

$$(p_{i_0,c_0} \vee \ldots \vee p_{i_0,c_l}) \wedge \ldots \wedge (p_{i_k,c_0} \vee \ldots \vee p_{i_k,c_l})$$

- At most one container per item

		Container			
		c_0	c_1	…	c_l
Item	i_0	p_{i_0,c_0}	p_{i_0,c_1}	…	p_{i_0,c_l}
	i_1	p_{i_1,c_0}	p_{i_1,c_1}	…	p_{i_1,c_l}
	…	…	…	…	…
	i_k	p_{i_k,c_0}	p_{i_k,c_1}	…	p_{i_k,c_l}

Generic constraints

- Consider 3 containers $(1,2,3)$ and 2 items (a, b)
- At least one container per item

$$(p_{a,1} \vee p_{a,2} \vee p_{a,3}) \wedge (p_{b,1} \vee p_{b,2} \vee p_{b,3})$$

- At most one container per item

$$\neg(p_{a,1} \wedge p_{a,2}) \wedge \neg(p_{a,1} \wedge p_{a,3}) \wedge \neg(p_{a,2} \wedge p_{a,3})$$
$$\wedge$$
$$\neg(p_{b,1} \wedge p_{b,2}) \wedge \neg(p_{b,1} \wedge p_{b,3}) \wedge \neg(p_{b,2} \wedge p_{b,3})$$

		Container		
		1	2	3
Item	*a*	$p_{a,1}$	$p_{a,2}$	$p_{a,3}$
	b	$p_{b,1}$	$p_{b,2}$	$p_{b,3}$

Generic constraints

- Consider 3 containers $(1,2,3)$ and 2 items (a, b)
- At least one container per item

$$(p_{a,1} \vee p_{a,2} \vee p_{a,3}) \wedge (p_{b,1} \vee p_{b,2} \vee p_{b,3})$$

- At most one container per item

$$(\neg p_{a,1} \vee \neg p_{a,2}) \wedge (\neg p_{a,1} \vee \neg p_{a,3}) \wedge (\neg p_{a,2} \vee \neg p_{a,3})$$
$$\wedge$$
$$(\neg p_{b,1} \vee \neg p_{b,2}) \wedge (\neg p_{b,1} \vee \neg p_{b,3}) \wedge (\neg p_{b,2} \vee \neg p_{b,3})$$

		Container		
		1	2	3
Item	*a*	$p_{a,1}$	$p_{a,2}$	$p_{a,3}$
	b	$p_{b,1}$	$p_{b,2}$	$p_{b,3}$

Generic constraints

- Consider 3 containers $(1,2,3)$ and *k* items $(i_0, \ldots, i_k)$
- At least one container per item

$$(p_{i_0,1} \vee p_{i_0,2} \vee p_{i_0,3}) \wedge \ldots \wedge (p_{i_k,1} \vee p_{i_k,2} \vee p_{i_k,3})$$

- At most one container per item

$$(\neg p_{i_0,1} \vee \neg p_{i_0,2}) \wedge (\neg p_{i_0,1} \vee \neg p_{i_0,3}) \wedge (\neg p_{i_0,2} \vee \neg p_{i_0,3})$$
$$\wedge \ldots \wedge$$
$$(\neg p_{i_k,1} \vee \neg p_{i_k,2}) \wedge (\neg p_{i_k,1} \vee \neg p_{i_k,3}) \wedge (\neg p_{i_k,2} \vee \neg p_{b,3})$$

		Container		
		1	2	3
Item	i_0	$p_{i_0,1}$	$p_{i_0,2}$	$p_{i_0,3}$
	i_1	$p_{i_1,1}$	$p_{i_1,2}$	$p_{i_1,3}$
	…	…	…	…
	i_k	$p_{i_k,1}$	$p_{i_k,2}$	$p_{i_k,3}$

Generic constraints

- Consider *l* containers ($c_0, \ldots, c_l$) and *k* items ($a_0, \ldots, a_k$)
- At least one container per item

$$(p_{i_0,c_0} \vee \ldots \vee p_{i_0,c_l}) \wedge \ldots \wedge (p_{i_k,c_0} \vee \ldots \vee p_{i_k,c_l})$$

- At most one container per item

$$(\neg p_{i_0,c_0} \vee \neg p_{i_0,c_1}) \wedge (\neg p_{i_0,c_0} \vee \neg p_{i_0,c_2}) \wedge \ldots \wedge (\neg p_{i_0,c_0} \vee \neg p_{i_0,c_l})$$
$$\wedge \ldots \wedge$$
$$(\neg p_{i_0,c_x} \vee \neg p_{i_0,c_{x+1}}) \wedge (\neg p_{i_0,c_x} \vee \neg p_{i_0,c_{x+2}}) \wedge \ldots \wedge (\neg p_{i_0,c_x} \vee \neg p_{i_0,c_l})$$
$$\wedge \ldots \wedge$$
$$(\neg p_{i_k,c_x} \vee \neg p_{i_k,c_{x+1}}) \wedge (\neg p_{i_k,c_x} \vee \neg p_{i_k,c_{x+2}}) \wedge \ldots \wedge (\neg p_{i_k,c_x} \vee \neg p_{i_k,c_l})$$

		Container			
		c_0	c_1	…	c_l
Item	i_0	p_{i_0,c_0}	p_{i_0,c_1}	…	p_{i_0,c_l}
	i_1	p_{i_1,c_0}	p_{i_1,c_1}	…	p_{i_1,c_l}
	…	…	…	…	…
	i_k	p_{i_k,c_0}	p_{i_k,c_1}	…	p_{i_k,c_l}

Checking assertions

- After adding the constraints describing the allocation problem we can check the validity of additional assertions
- To check if ϕ is valid add $\neg\phi$ as a constraint and check for satisfiability
  - If it is satisfiable, then $\neg\phi$ holds for some assignment, so ϕ is not valid

Placement of guests

- We have three chairs in a row and need to place Anne, Susan and Peter
  - Anne does not want to sit next to Peter
  - Anne does not want to sit in the left chair
  - Susan does not want to sit to the left of Peter
- Implicit generic constraints
  - Everyone must be sited in a chair (at least one chair per guest)
  - No more than one guest per chair (at most one guest per chair)

Variables

		Chair		
		Left	Center	Right
Guest	Anne	$x_{a,l}$	$x_{a,c}$	$x_{a,r}$
	Susan	$x_{s,l}$	$x_{s,c}$	$x_{s,r}$
	Peter	$x_{p,l}$	$x_{p,c}$	$x_{p,r}$

At least one chair per guest

$$(x_{a,l} \vee x_{a,c} \vee x_{a,r}) \wedge (x_{s,l} \vee x_{s,c} \vee x_{s,r}) \wedge (x_{p,l} \vee x_{p,c} \vee x_{p,r})$$

At most one guest per chair

$$(\neg x_{a,l} \vee \neg x_{s,l}) \wedge (\neg x_{a,l} \vee \neg x_{p,l}) \wedge (\neg x_{s,l} \vee \neg x_{p,l})$$
$$\wedge$$
$$(\neg x_{a,c} \vee \neg x_{s,c}) \wedge (\neg x_{a,c} \vee \neg x_{p,c}) \wedge (\neg x_{s,c} \vee \neg x_{p,c})$$
$$\wedge$$
$$(\neg x_{a,r} \vee \neg x_{s,r}) \wedge (\neg x_{a,r} \vee \neg x_{p,r}) \wedge (\neg x_{s,r} \vee \neg x_{p,r})$$

Problem specific constraints

- Anne does not want to sit next to Peter

$$(x_{a,c} \rightarrow (\neg x_{p,l} \wedge \neg x_{p,r})) \wedge ((x_{a,l} \vee x_{a,r}) \rightarrow \neg x_{p,c})$$

- Anne does not want to sit in the left chair

$$\neg x_{a,l}$$

- Susan does not want to sit to the left of Peter

$$(x_{p,r} \rightarrow \neg x_{s,c}) \wedge (x_{p,c} \rightarrow \neg x_{s,l})$$

Finding a placement

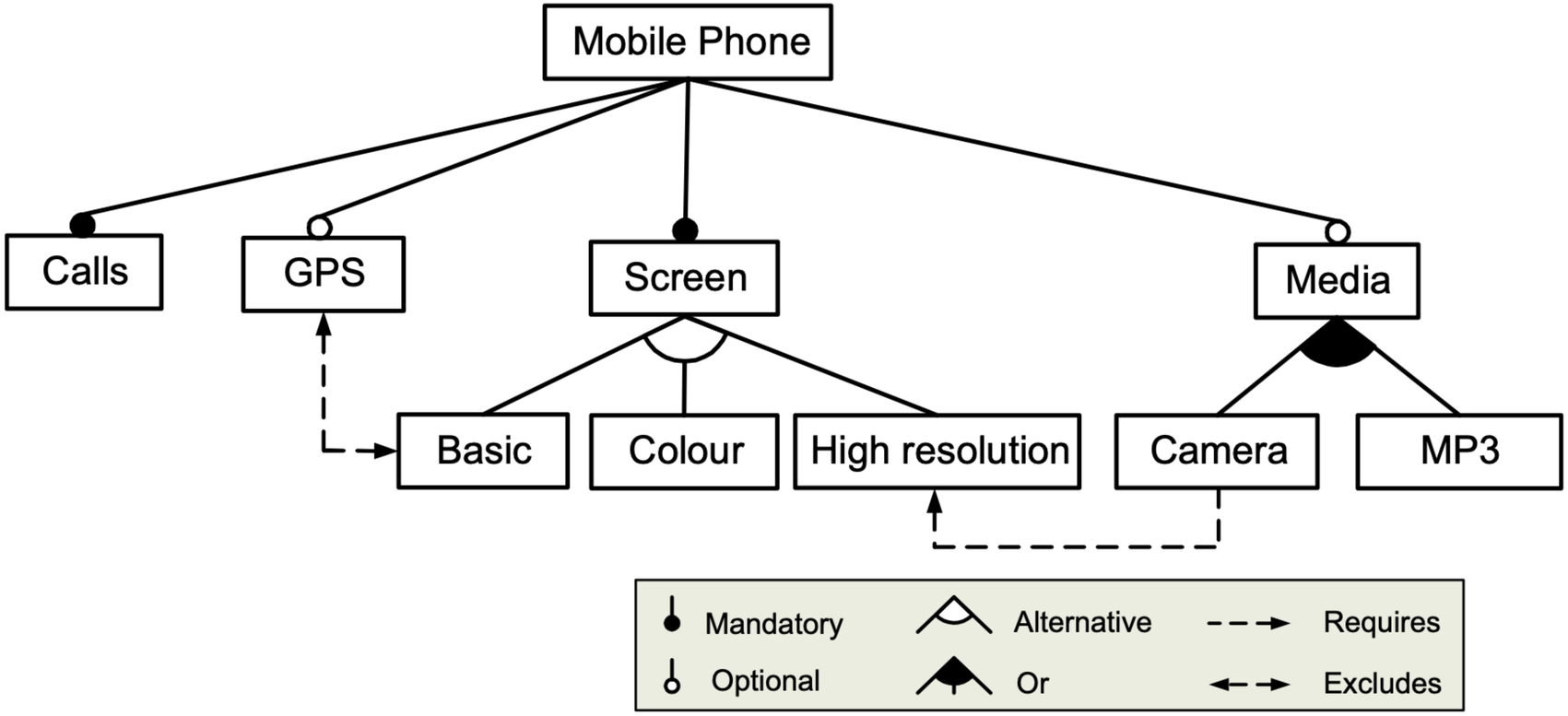
- A placement is an assignment to variables for which the constraints hold
  - SAT solving to show it is satisfiable: finding a $\mathscr{A}$ under which it holds
- We can do this manually, there are “only” $2^9 = 512$ possible assignments
  - Only one solution: $x_{p,l}, x_{a,c}, x_{s,r}$ true, all other variables false
- Later we will revisit this with automated SAT solvers

Feature model analysis

Variability modelling

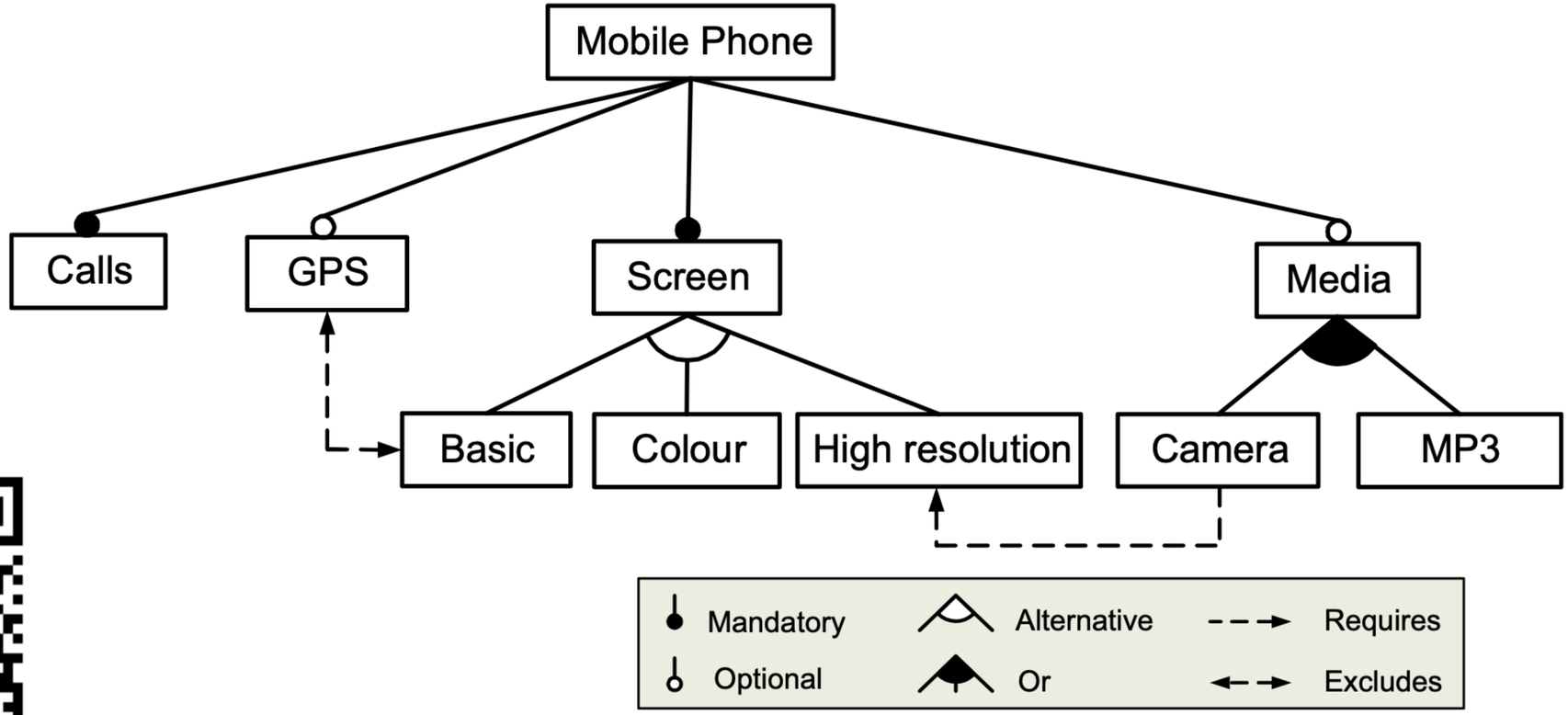
- A **software product line** (SPL) is a family of software products
- Each **product** or **variant** supports different features
- A **feature** is an increment in program functionality
- A **feature model** is a compact representation of the variability of a SPL

Feature models



https://doi.org/10.1016/j.is.2010.01.001

How many phone variants exist?



Feature model analysis

- Relevant analyses of a feature model
  - check if it is not void (there are products)
  - check if a feature is “dead” (no product can implement it)
  - check if a feature is “core” (all products implement it)
  - count how many different products exist
- All these analyses can be easily implemented using SAT solving

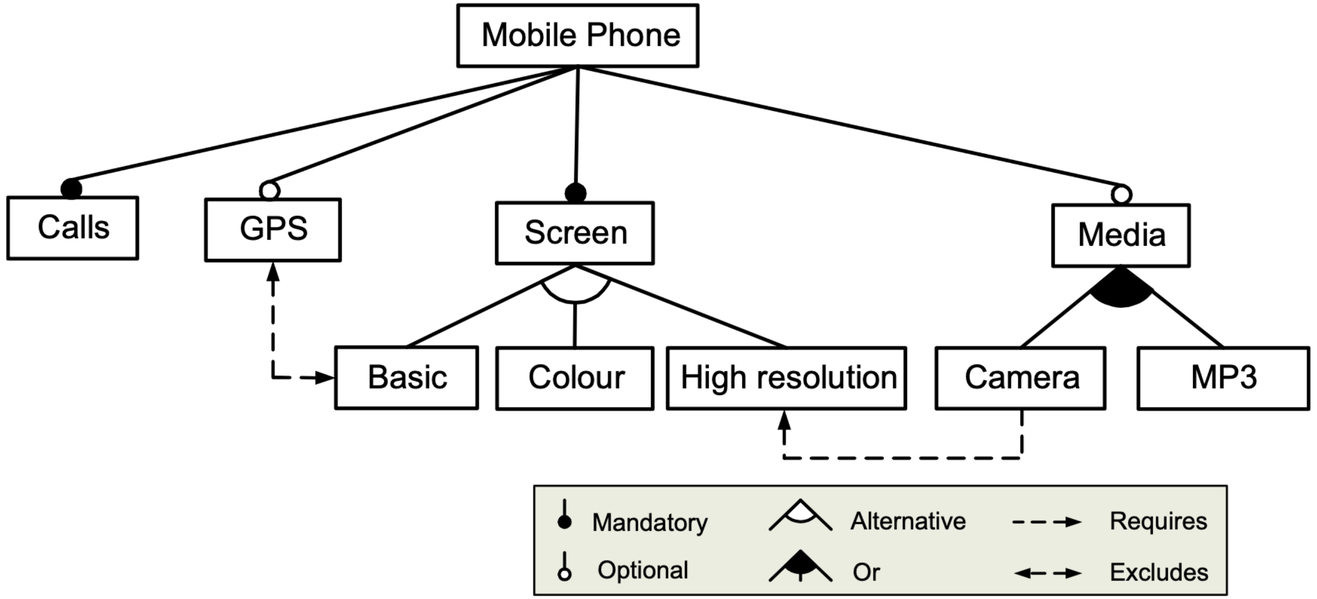
Feature model semantics

- A feature model can be encoded with a propositional formula ϕ
- The presence of each feature corresponds to a propositional variable
- Each feature model primitive corresponds to a conjunct of ϕ

Feature model semantics

r is the root feature	r
f_1 mandatory sub-feature of f	$f_1 \leftrightarrow f$
f_1 optional sub-feature of f	$f_1 \rightarrow f$
$f_1 \ldots f_n$ or sub-features of f	$f_1 \vee \ldots \vee f_n \leftrightarrow f$
$f_1 \ldots f_n$ xor sub-features of f	$(f_1 \vee \ldots \vee f_n \leftrightarrow f) \wedge \neg(f_i \wedge f_{i+1}) \wedge \ldots \wedge \neg(f_i \wedge f_n)$
f_1 requires f_2	$f_1 \rightarrow f_2$
f_1 excludes f_2	$\neg(f_1 \wedge f_2)$

Feature model semantics



Phone

Calls ↔ Phone

GPS → Phone

Screen ↔ Phone

Media → Phone

(Basic ∨ Color ∨ Highres) ↔ Screen

¬(Basic ∧ Color) ∧ ¬(Basic ∧ Highres) ∧ ¬(Color ∧ Highres)

(Camera ∨ MP3) ↔ Media

¬(GPS ∧ Basic)

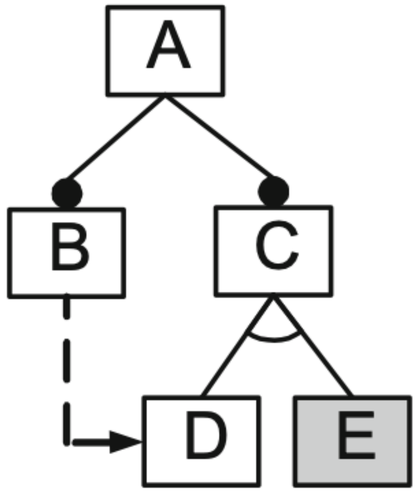
Camera → Highres

Non voidness

- Given a formula ϕ that encodes the semantics of a feature model
- To check if the feature model is not void
  - check if ϕ is satisfiable

Dead feature

- Given a formula ϕ that encodes the semantics of a feature model
- To check if feature f is dead
  - check if $\phi \wedge f$ is unsatisfiable



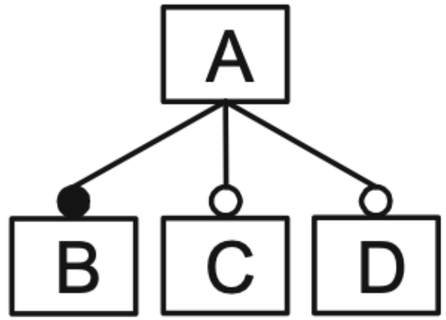
Core feature

- Given a formula ϕ that encodes the semantics of a feature model
- To check if feature f is core
  - check if $\phi \wedge \neg f$ is unsatisfiable

Counting products

- Given a formula ϕ that encodes the semantics of a feature model
- To count the number of products count the number of iterations of the following cycle
  - Repeat while ϕ is satisfiable
    - extract an assignment $\mathcal{A}$ under which ϕ holds
    - convert $\mathcal{A}$ to a formula $\overline{\mathcal{A}}$ that holds only for $\mathcal{A}$
    - add $\neg\overline{\mathcal{A}}$ as a new conjunct of ϕ

Counting products



$$\phi_0 \equiv A \wedge (A \leftrightarrow B) \wedge (C \rightarrow A) \wedge (D \rightarrow A)$$

$$\mathscr{A}_1 \equiv \{A \mapsto 1, B \mapsto 1, C \mapsto 0, D \mapsto 0\}$$

$$\phi_1 \equiv A \wedge (A \leftrightarrow B) \wedge (C \rightarrow A) \wedge (D \rightarrow A) \wedge \neg(A \wedge B \wedge \neg C \wedge \neg D)$$

$$\mathscr{A}_2 \equiv \{A \mapsto 1, B \mapsto 1, C \mapsto 1, D \mapsto 0\}$$

$$\phi_2 \equiv A \wedge (A \leftrightarrow B) \wedge (C \rightarrow A) \wedge (D \rightarrow A) \wedge \neg(A \wedge B \wedge \neg C \wedge \neg D) \wedge \neg(A \wedge B \wedge C \wedge \neg D)$$

$$\mathscr{A}_3 \equiv \{A \mapsto 1, B \mapsto 1, C \mapsto 0, D \mapsto 1\}$$

$$\phi_3 \equiv \ldots$$

$$\mathscr{A}_4 \equiv \{A \mapsto 1, B \mapsto 1, C \mapsto 1, D \mapsto 1\}$$

$$\phi_4 \equiv \ldots$$