Mastering Alloy

Alcino Cunha

Subset signatures

Subset signatures

- An arbitrary subset of a signature can be declared with keyword **`in`** instead of **`extends`**
- Subset signatures are not necessarily disjoint
- A signature can be declared as a subset of more than one signature
- Subset signatures cannot be extended
- Subset signatures can be used to simulate multiple-inheritance
- Atoms belonging to subset signatures are labelled in the visualiser

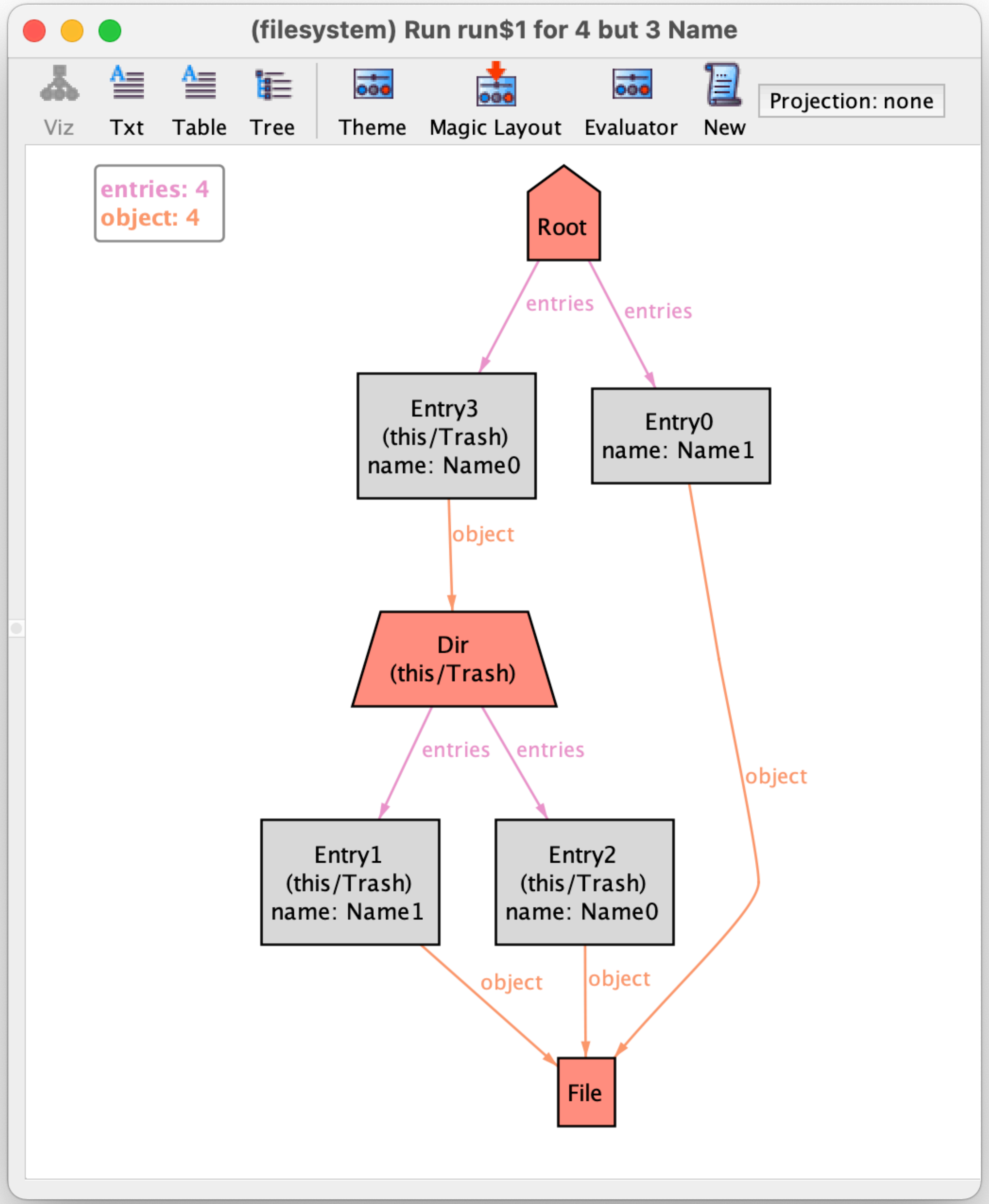
File-system

```
abstract sig Object {}
sig Dir extends Object {
  entries : set Entry
}
sig File extends Object {}
one sig Root extends Dir {}
sig Entry {
  object : one Object,
  name   : one Name
}
sig Name {}
```

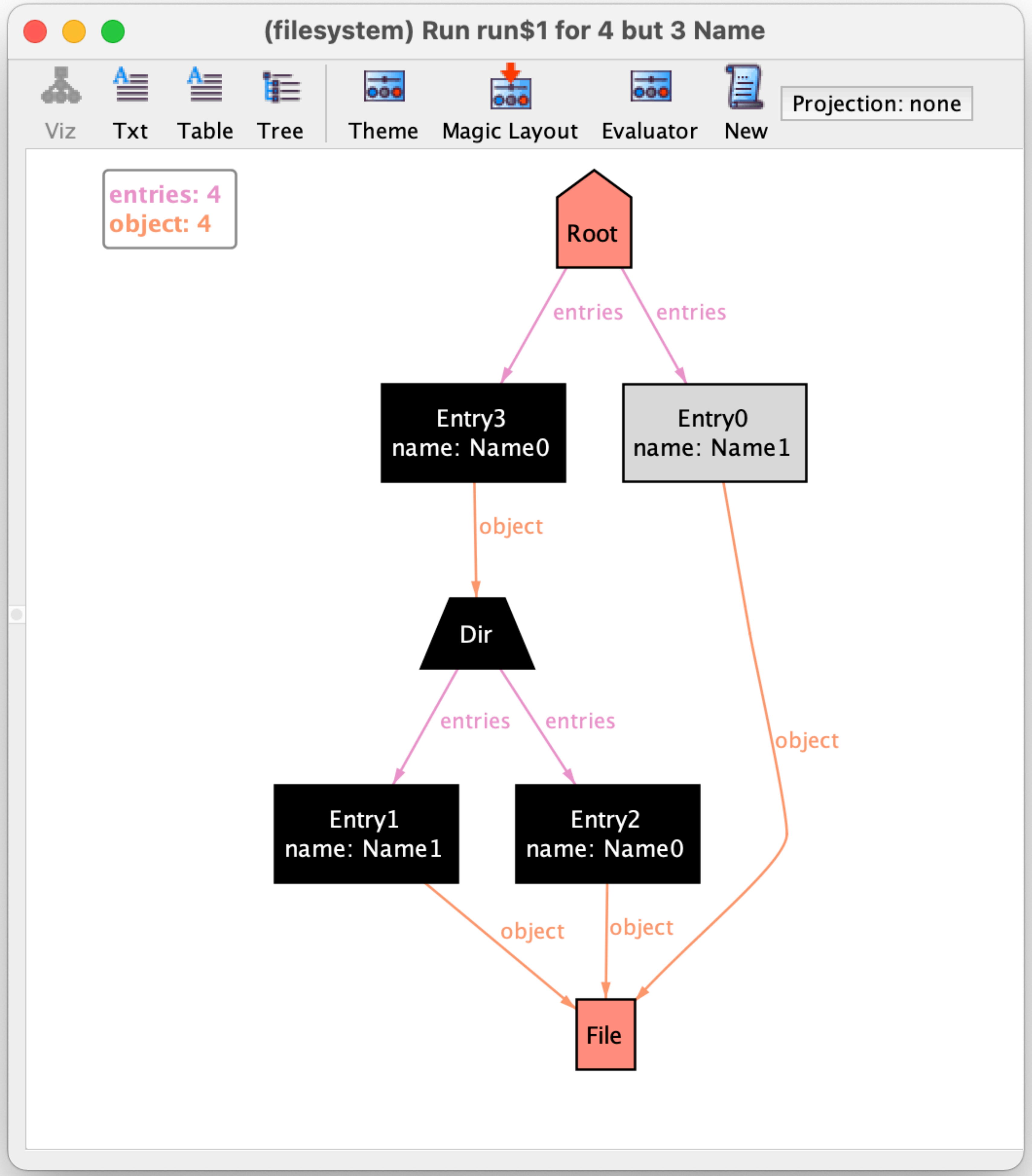
Subset example

```
sig Trash in Object+Entry {}

fact {
  // The root cannot be trashed
  Root not in Trash
  // All other objects are trashed iff all entries
  // that point to them are trashed
  all o : Object-Root | o in Trash iff object.o in Trash
  // If a directory is trashed all its entries are trashed
  all d : Dir & Trash | d.entries in Trash
}
```

Visualising subsets



Visualising subsets



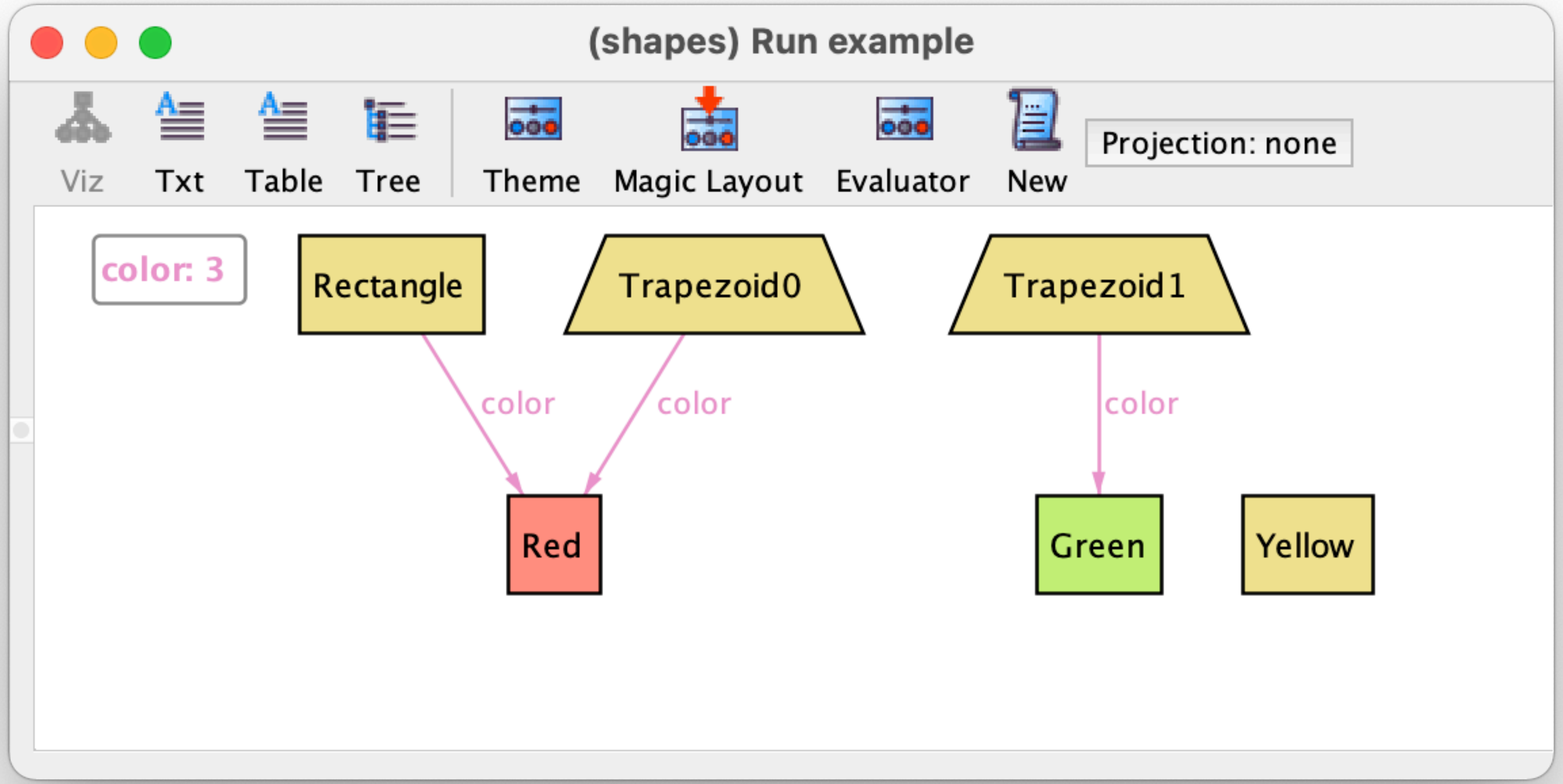
Subset example

```
abstract sig Shape {
  color : one Color
}
sig Rectangle, Trapezoid extends Shape {}

abstract sig Color {}
one sig Green, Red, Yellow extends Color {}
```

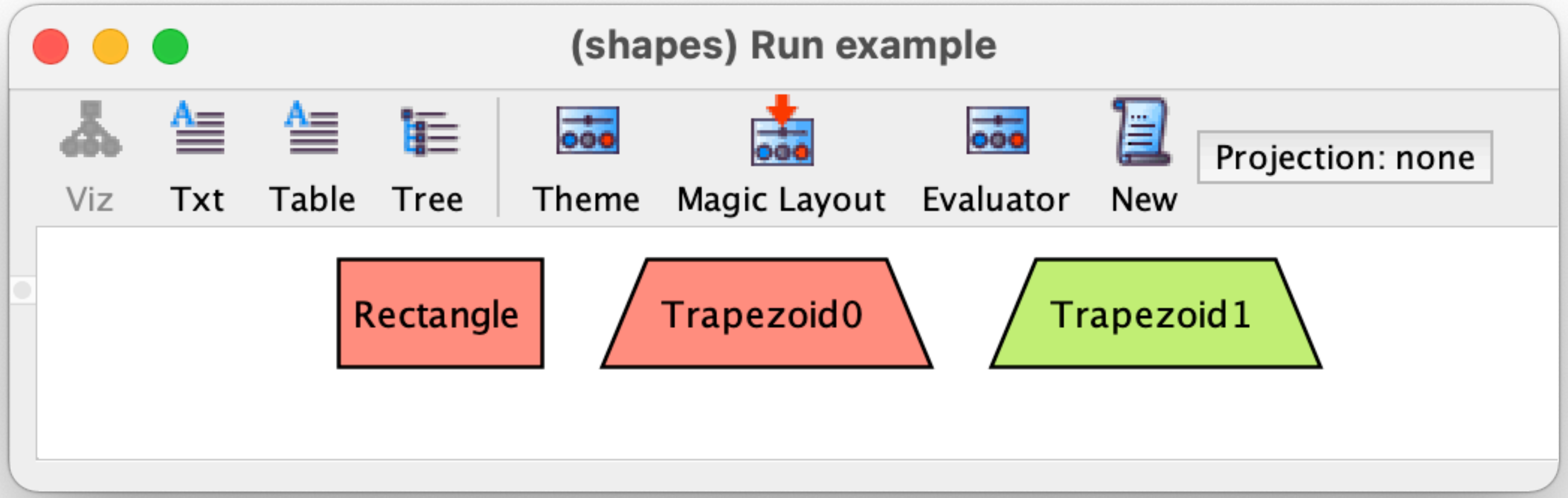
Subset example

```
abstract sig Shape {}
sig Rectangle, Trapezoid extends Shape {}

sig Green, Red, Yellow in Shape {}

fact {
  Shape = Green+Red+Yellow
  no Green & Red
  no Red & Yellow
  no Green & Yellow
}
```

Visualising subsets

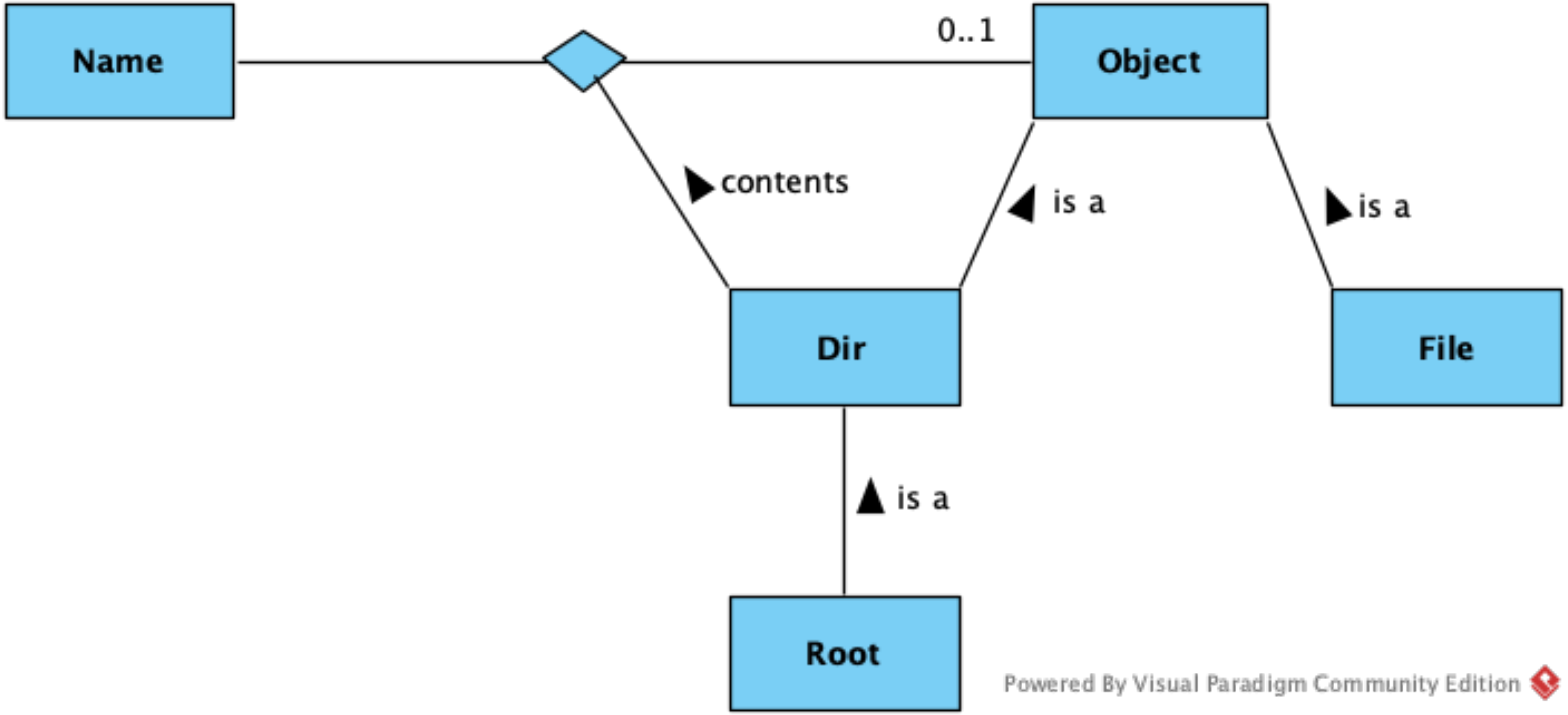


Visualising subsets



N-ary relations

N-ary relationships *a la* UML



Ternary relation example

```
abstract sig Object {}
sig Dir extends Object {
  contents : Name -> lone Object
}
sig File extends Object {}
one sig Root extends Dir {}
sig Name {}
```

Composition

Root
Root

contents		
Root	Name0	Dir
Root	Name1	File
Dir	Name0	File
Dir	Name1	File

Composition

Root
Root

contents		
Root	Name0	Dir
Root	Name1	File
Dir	Name0	File
Dir	Name1	File

`Root . contents`

Composition

Root
Root

contents		
Root	Name0	Dir
Root	Name1	File
Dir	Name0	File
Dir	Name1	File

Root . contents	
Name0	Dir

Composition

Root
Root

contents		
Root	Name0	Dir
Root	Name1	File
Dir	Name0	File
Dir	Name1	File

Root . contents	
Name0	Dir
Name1	File

Composition

Root
Root

contents		
Root	Name0	Dir
Root	Name1	File
Dir	Name0	File
Dir	Name1	File

Root . contents	
Name0	Dir
Name1	File

Ternary relation example

```
fact {
   // All objects except the root are contained in at least one directory
   all o : Object - Root | some contents.o
   no contents.Root

   // All directories are contained in at most one directory
   all d : Dir | lone contents.d

   // A directory cannot be contained in itself
   all d : Dir | d not in d.^(???)
}
```

Comprehension

$$\{\ x_1 : A_1, \ldots, x_n : A_n \ \mid\ \phi\ \}$$

$$\{\ x_1 : A_1, \ldots, x_n : A_n \ \mid\ \phi\ \}\ (y_1, \ldots, y_n)$$

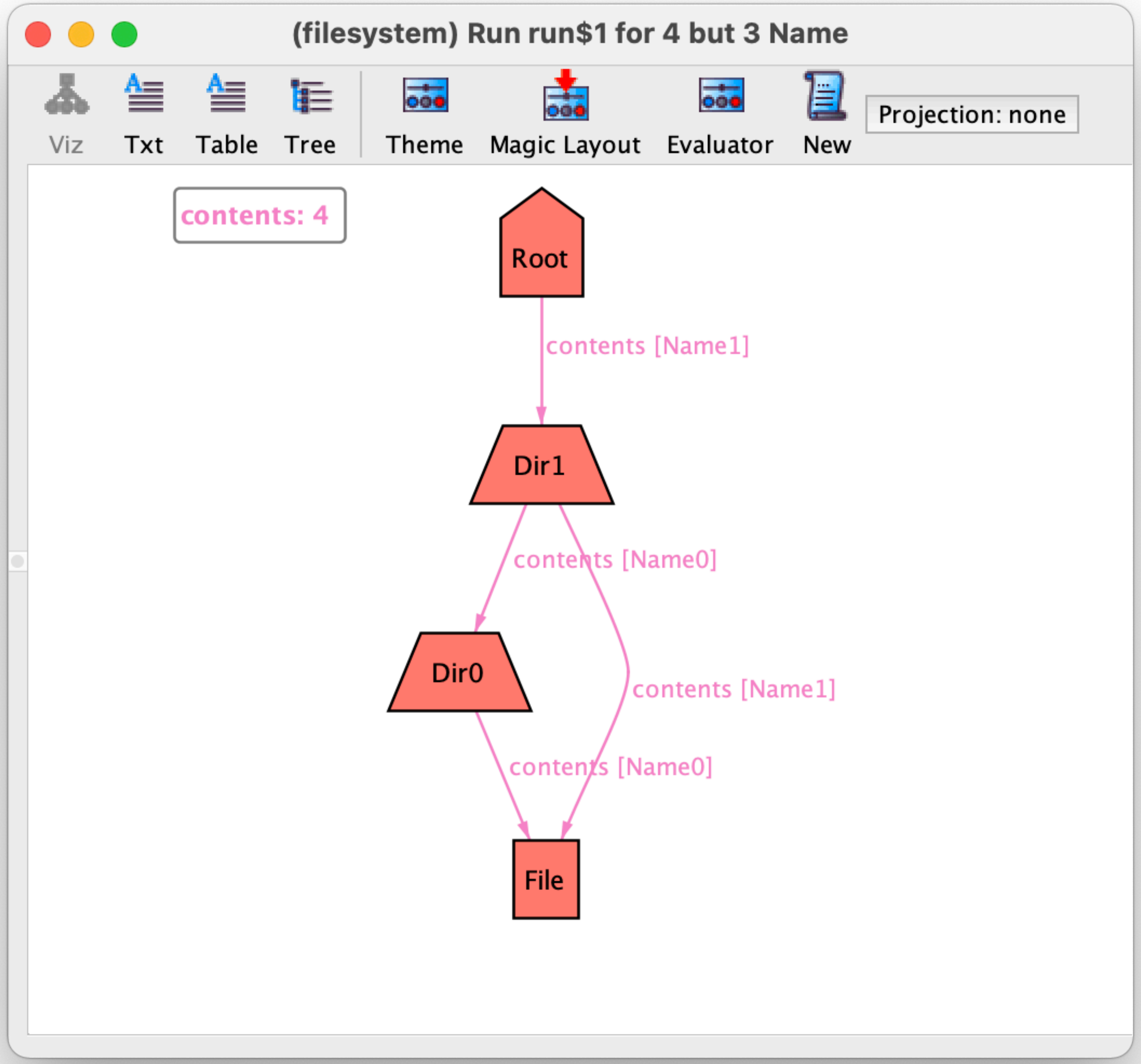
$$\leftrightarrow$$

$$A_1(y_1) \wedge \ldots \wedge A_n(y_n)\ \wedge\ \phi[x_1 \leftarrow y_1, \ldots, x_n \leftarrow y_n]$$

Ternary relation example

```
fact {
   // All objects except the root are contained in at least one directory
   all o : Object - Root | some contents.o
   no contents.Root

   // All directories are contained in at most one directory
   all d : Dir | lone contents.d

   // A directory cannot be contained in itself
   all d : Dir | d not in d.^({d : Dir, o : Object | some d.contents.o})
}
```

Visualising N-ary relations



Overloading

Overloading

- Fields in disjoint signatures can be *overloaded* (have the same name)
- *Ambiguity* errors may occur

Overloading example

```
abstract sig Object {}
sig Dir extends Object {
   contents : set Entry
}
sig File extends Object {}
one sig Root extends Dir {}
sig Entry {
   contents : one Object,
   name     : one Name
}
sig Name {}
```

Overloading example

```
fact {
   // All objects except the root are contained in at least one entry
   all o : Object - Root | some contents.o
   no contents.Root

   // All directories are contained in at most one entry
   all d : Dir | lone contents.d

   // Different entries in a directory must have different names
   all d : Dir, n : Name | lone (d.contents & name.n)

   // A directory cannot be contained in itself
   all d : Dir | d not in d.^(contents.contents)
}
```

Ambiguity errors

```
run { some contents }
```

```
A type error has occurred:
This name is ambiguous due to multiple matches:
field this/Dir <: contents
field this/Entry <: contents
```

Resolving ambiguities

```
run { some d : Dir | some d.contents }

run { some contents & Dir->Entry }

run { some Dir <: contents }
```

Predicates and functions

Predicates

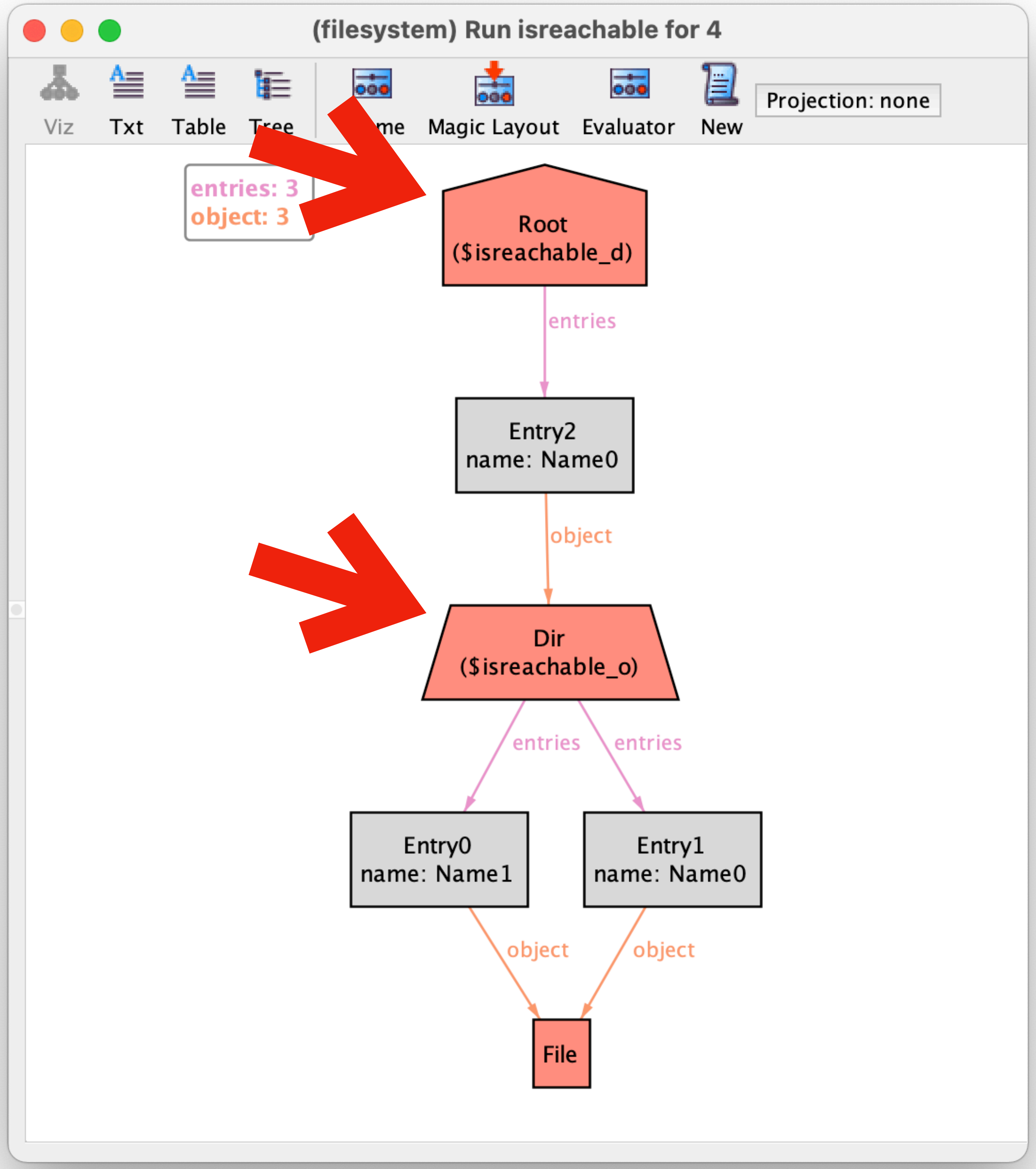
- *Predicates* are parametrised reusable constraints
    - Can also be derived propositions (without arguments)
    - Parameters can be arbitrary relations
- Only hold when invoked in a fact, command, or other predicates
- Recursive definitions are not allowed
- Run commands can directly ask for an instance satisfying a predicate
    - Atoms instantiating the parameters are shown in the visualiser

Predicate example

```
pred isreachable [d : Dir, o : Object] {
  o in d.^(entries.object)
}

fact {
  // A directory cannot be contained in itself
  all d : Dir | not isreachable[d,d]
}
```

Running a predicate

run isreachable **for** 4



Higher-order predicate example

```
pred acyclic [r : univ -> univ] {
  no ^r & iden
}

fact {
  // A directory cannot be contained in itself
  acyclic[entries.object]
}
```

Functions

- *Functions* are parametrised reusable expressions
  - Parameters can be arbitrary relations
- Functions without parameters can be used to define derived relations
  - These show up in the visualiser
- Recursive definitions are not allowed

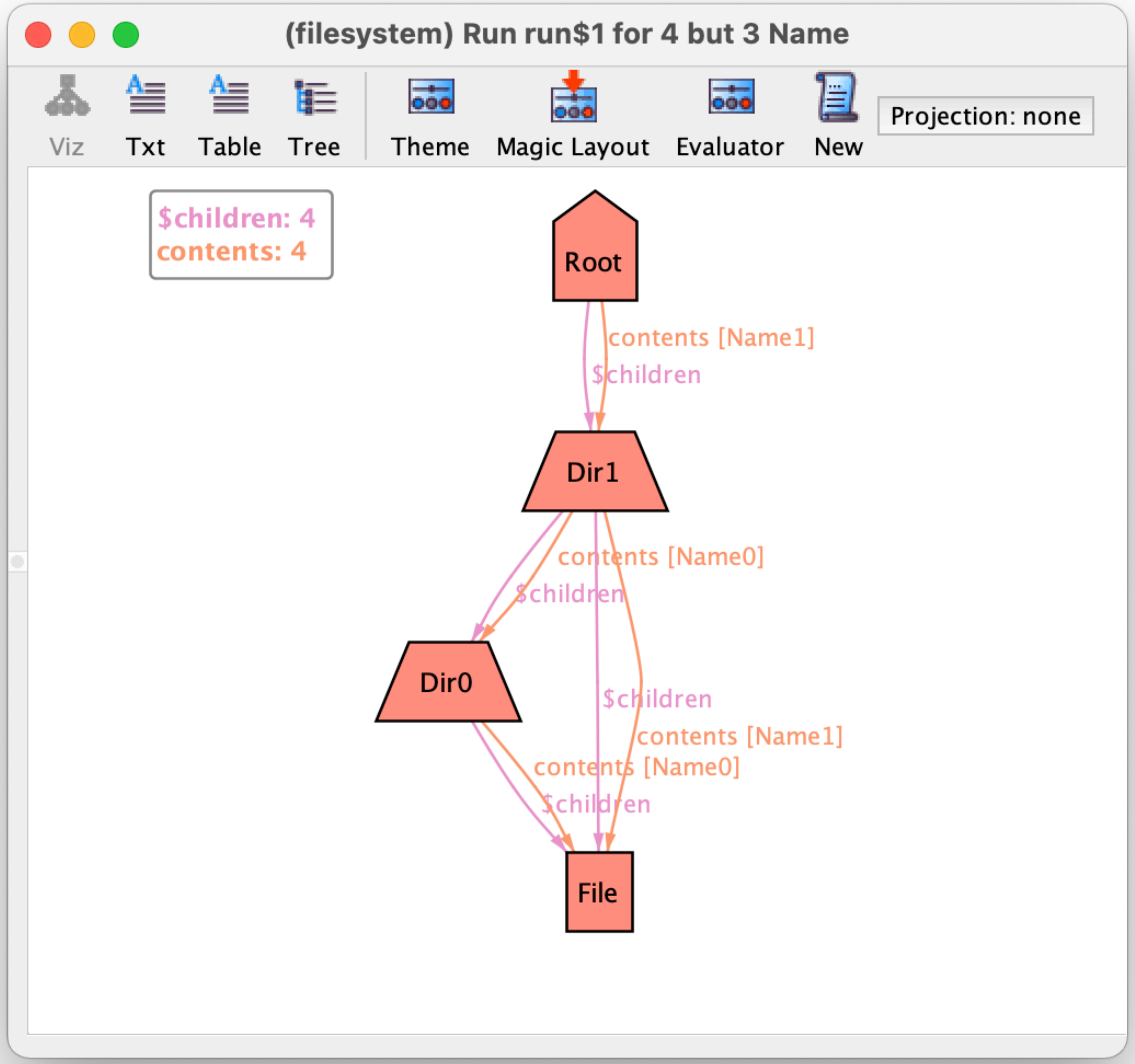
Function example

```
fun descendants [d : Dir] : set Object {
  d.^(entries.object)
}

fact {
  // A directory cannot be contained in itself
  all d : Dir | d not in descendants[d]
}
```

Derived relation example

```
sig Dir extends Object {
   contents : Name -> lone Object
}

fun children : Dir -> Object {
   { d : Dir, o : Object | some d.contents.o }
}

fact {
   // A directory cannot be contained in itself
   all d : Dir | d not in d.^children
}
```

Visualising derived relations



Modules

Modules

- A model can be split into *modules*
- A module name is declared in the first line with keyword **`module`**
- A module can be imported with an **`open`** statement
- A module name must match the path of the corresponding file
- To disambiguate a call to an entity, the module name can be prepended
- An alias to a module name can be given with the **`as`** keyword in an **`open`** statement
- A module can be parametrised by one or more signatures

Module example

```
module relation

pred acyclic [r : univ -> univ] {
   no ^r & iden
}
```

Module example

```
open relation

fact {
  // A directory cannot be contained in itself
  acyclic[entries.object]
}
```

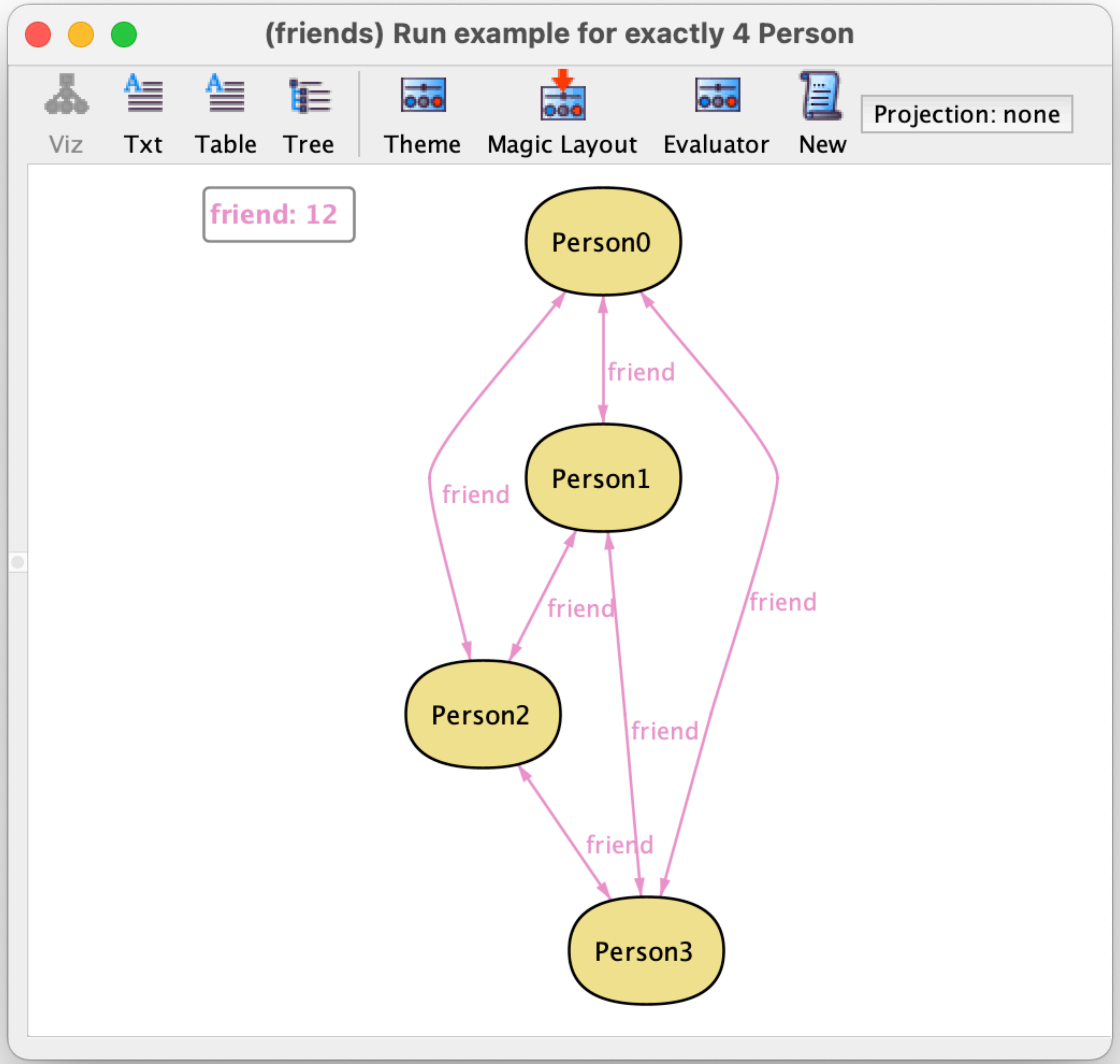
Parametrised module example

```
module graph[node]

pred complete[adj : node -> node] {
  all n : node | n.adj = node-n
}
```

Parametrised module example

```
open graph[Person]

sig Person {
  friend : set Person
}

fact { complete[friend] }
```

We are all friends



Predefined modules

`util/relations`	Useful functions and predicates for binary relations
`util/ternary`	Useful functions and predicates for ternary relations
`util/graph[A]`	Useful functions and predicates for graphs with nodes from signature *A*
`util/natural`	Natural numbers, including some arithmetic operations
`util/boolean`	Boolean type, including common logical connectives
`util/ordering[A]`	Imposes a total order on signature *A*

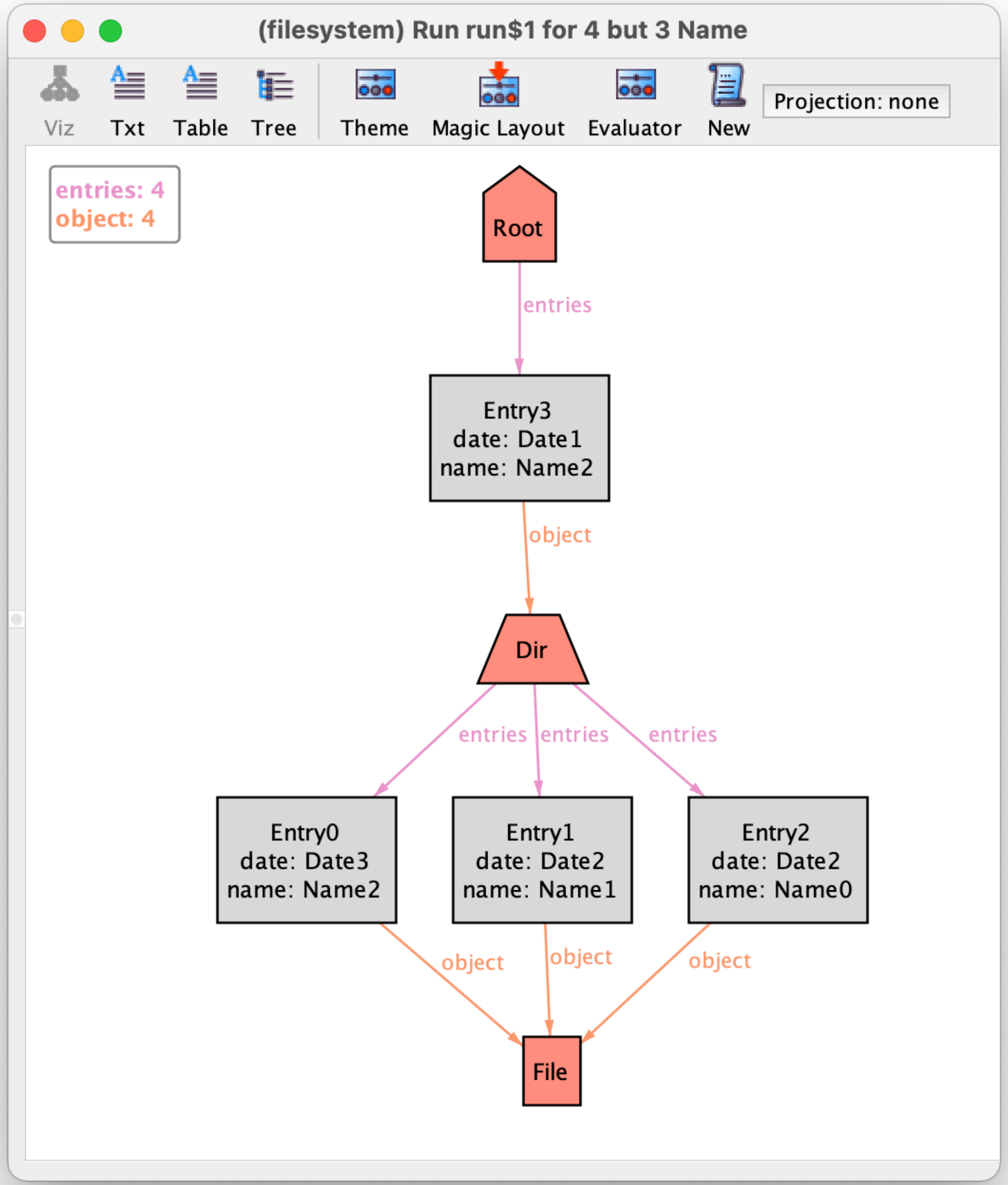
`util/ordering`

- Imposes a total order on the parameter signature
- For efficiency reasons the scope on that signature becomes exact
- Visualiser attempts to name atoms according to the order
- Many useful functions and relations, including
  - `next` and `prev` binary relations
  - `first` and `last` singleton sets
  - `lt`, `lte`, `gt`, and `gte` comparison predicates

`util/ordering` example

```
open util/ordering[Date]

sig Date {}
sig Entry {
   object : one Object,
   name   : one Name,
   date   : one Date
}

fact {
   // Entries inside a directory must have been created later
   all e : object.Dir, c : e.object.entries | lt[e.date, c.date]
}
```

util/ordering visualisation



Type system

File-system

```
abstract sig Object {}
sig Dir extends Object {
  contents : set Entry
}
sig File extends Object {}
one sig Root extends Dir {}
sig Entry {
  contents : one Object,
  name    : one Name
}
sig Name {}
```

Arity error

```
run { some name & File }
```

```
A type error has occurred:
& can be used only between 2 expressions of the same arity.
Left type = {this/Entry->this/Name}
Right type = {this/File}
```

Ambiguity error

```
run { some contents }
```

```
A type error has occurred:
This name is ambiguous due to multiple matches:
field this/Dir <: contents
field this/Entry <: contents
```

Irrelevance warning

```
run { some Dir.name }
```

```
Warning #1
The join operation here always yields an empty set.
Left type = {this/Dir}
Right type = {this/Entry->this/Name}
```

Type system

- The main goal of Alloy’s type system is to detect *irrelevant* expressions
- An expression is irrelevant if it can be replaced by **`none`**
- The same type system can be used to resolve overloading
  - An overloaded name is treated as the union of all respective relations
  - Only one of the overloaded relations must be relevant

Types

- The type of an expression is a set of tuples of atomic of *atomic* types
  - An atomic type is a signature that is not further extended
- For non abstract signatures we need a *reminder* type
  - The reminder contains all atoms not contained in one of the extensions
  - The reminder type of signature A is denoted as $\$A$
- The type of an expression is an upper-bound on its value
  - If the type of an expression is empty, the expression is irrelevant

Type inference

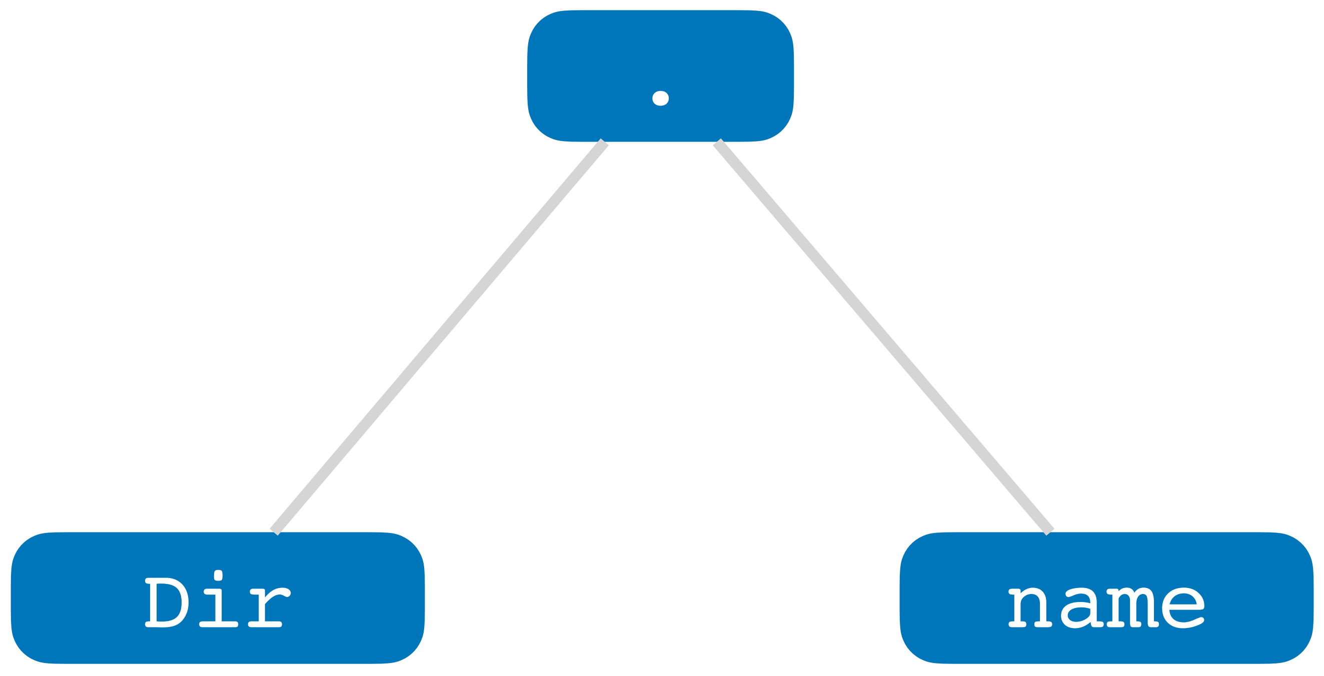
- Type inference is guided by the abstract syntax and works in two phases
  - A bottom-up phase computes the *bounding* types
  - A top-down phase refines these and computes the *relevance* types
- Unlike bounding types, relevance types depend on the context
  - The same expression in different formulas may have different types

Bounding type inference

- The bounding type of a signature or field is inferred from the respective declarations
- The bounding type of an expression is computed using the same relational operator applied to the bounding types of sub-expressions
  - This is possible because types are also relations

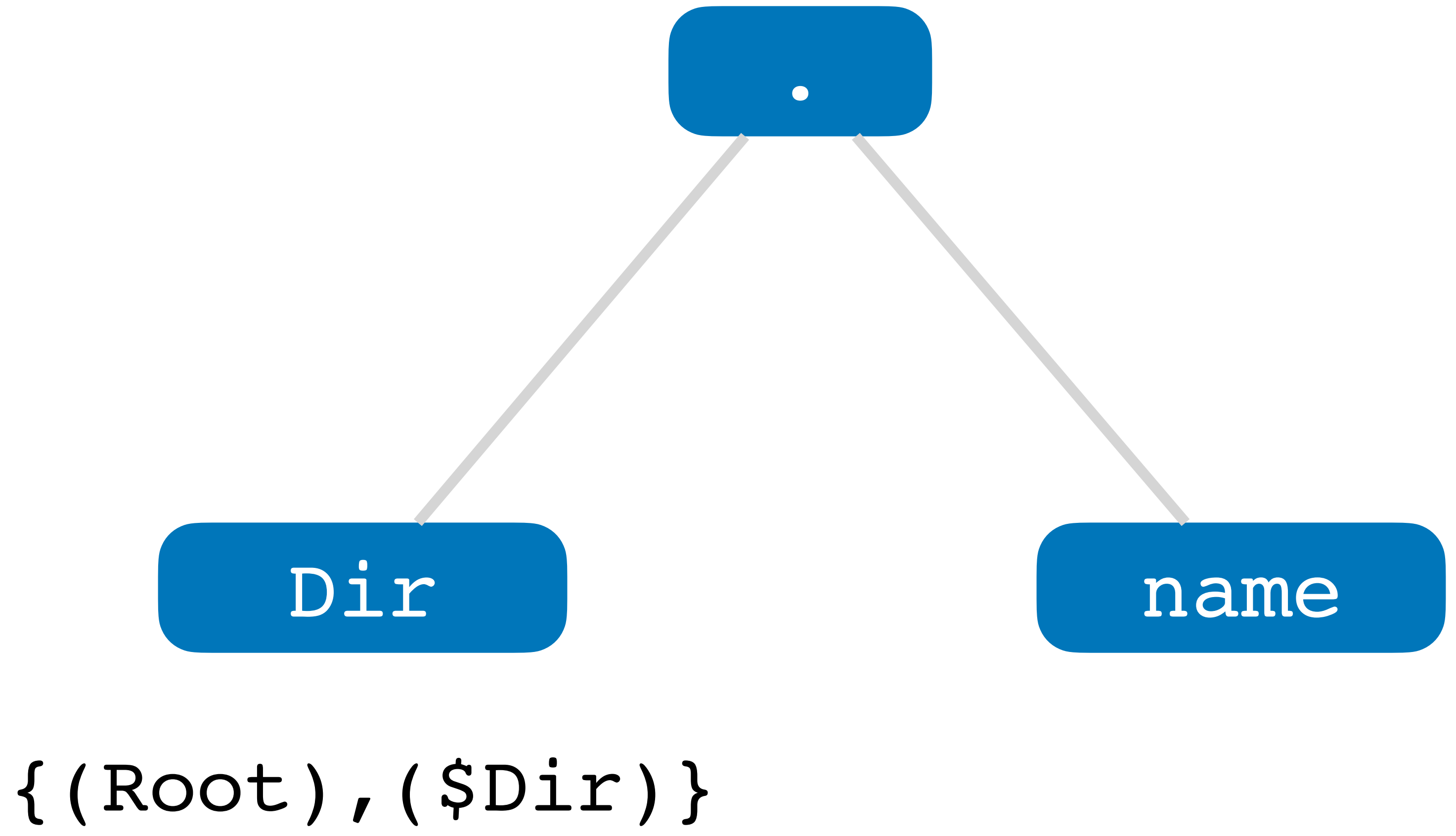
Bounding type inference

`Dir.name`



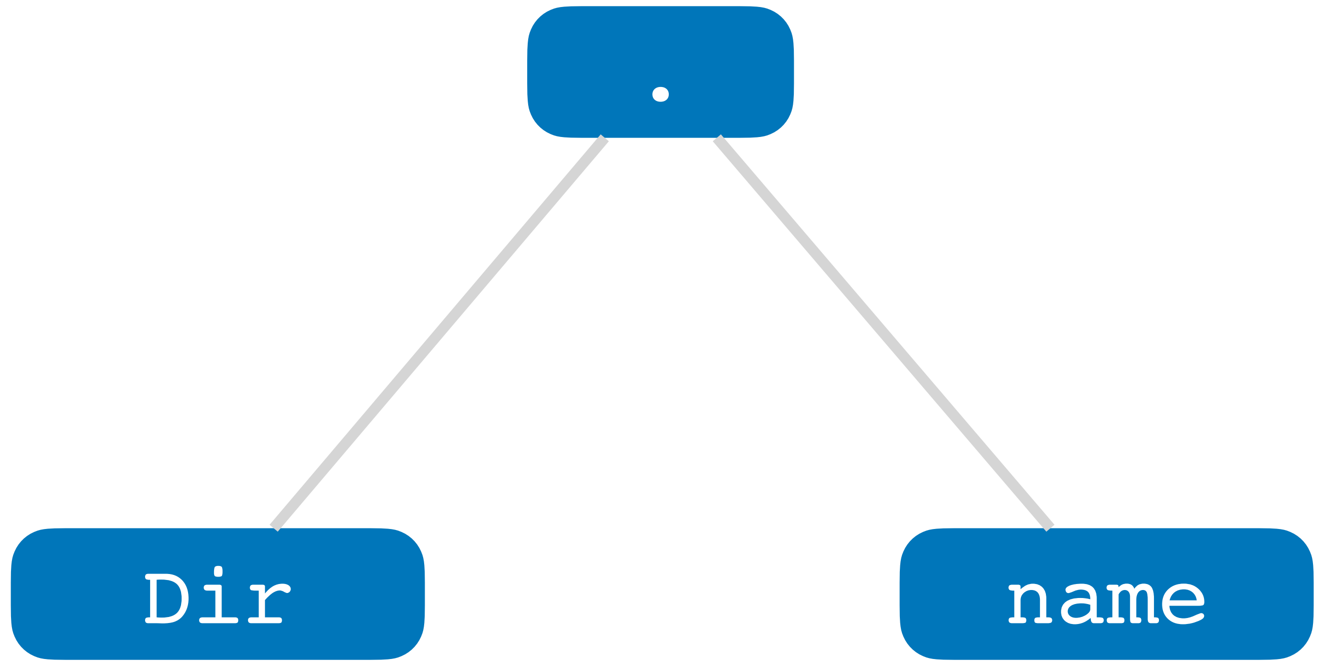
Bounding type inference

`Dir.name`



Bounding type inference

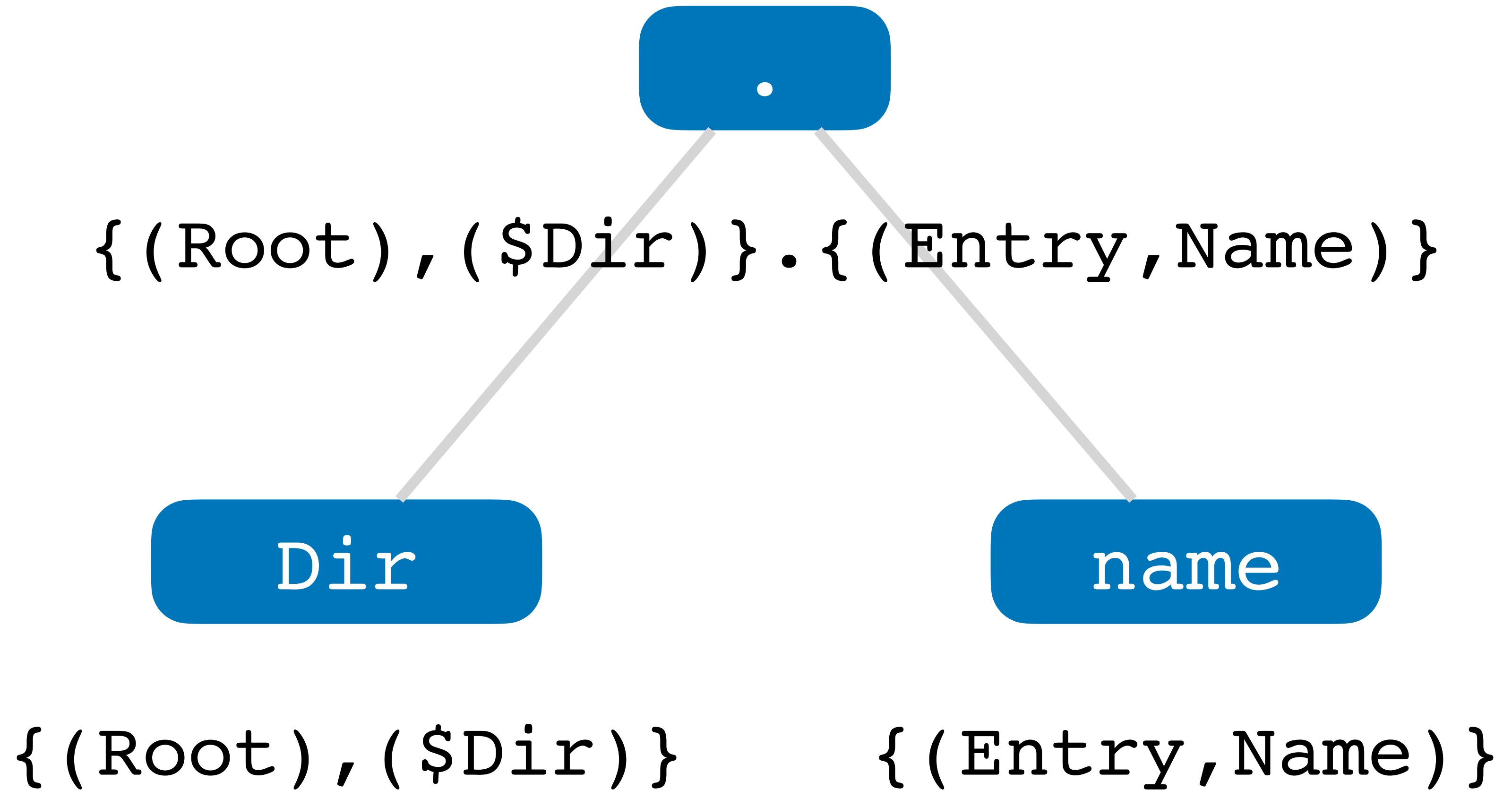
`Dir.name`



`{(Root),($Dir)}` `{(Entry,Name)}`

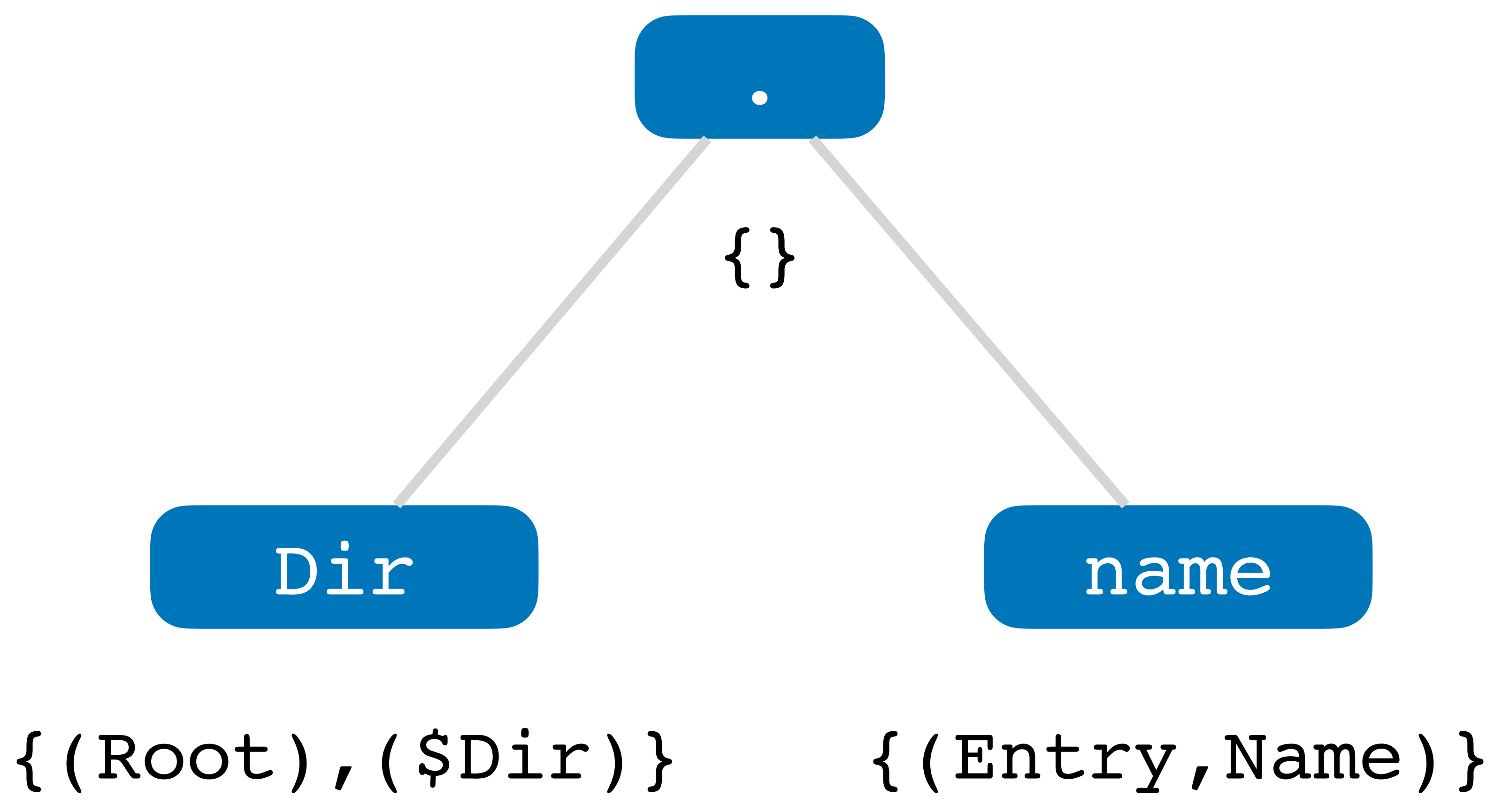
Bounding type inference

```
Dir.name
```



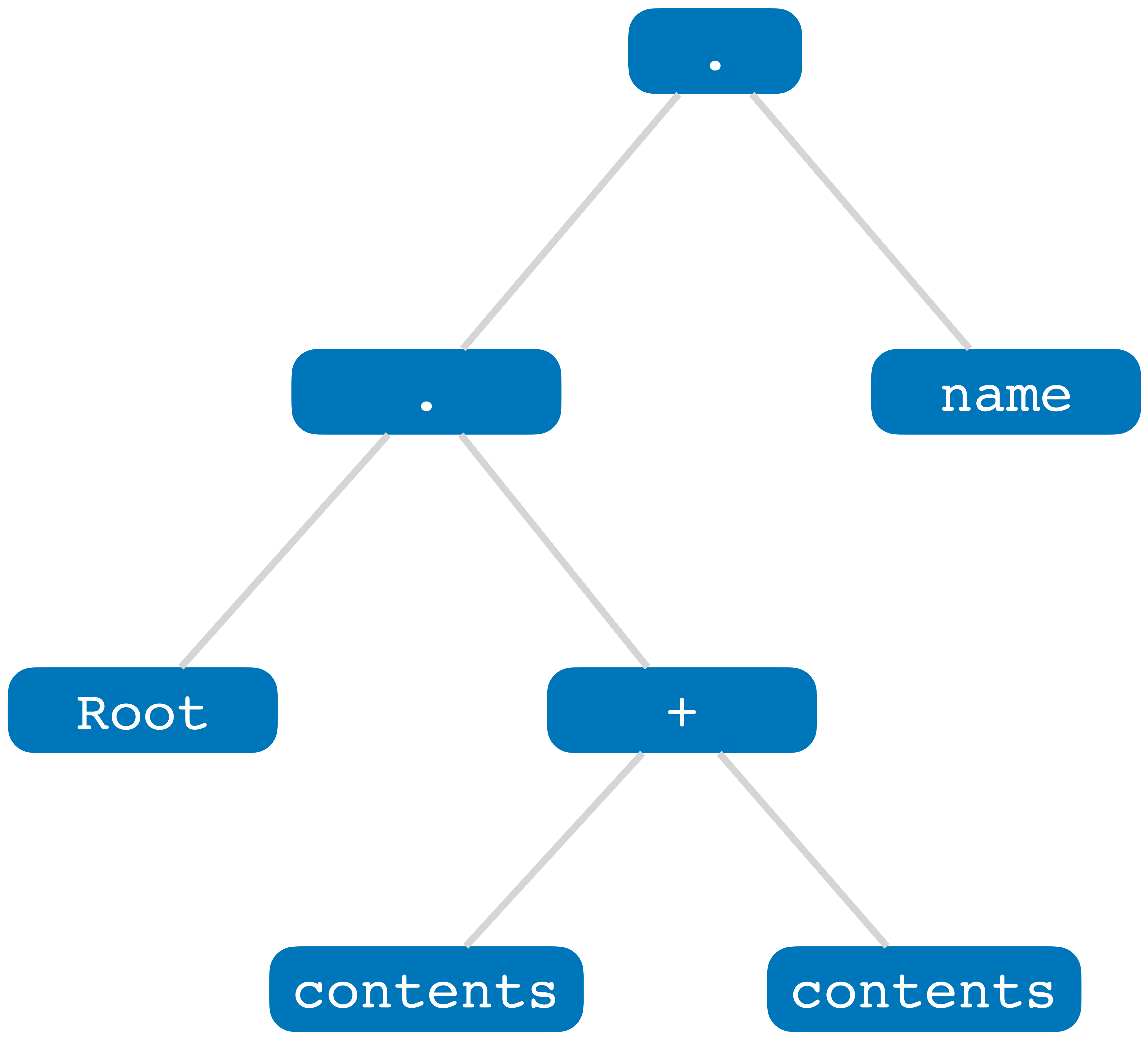
Bounding type inference

`Dir.name`



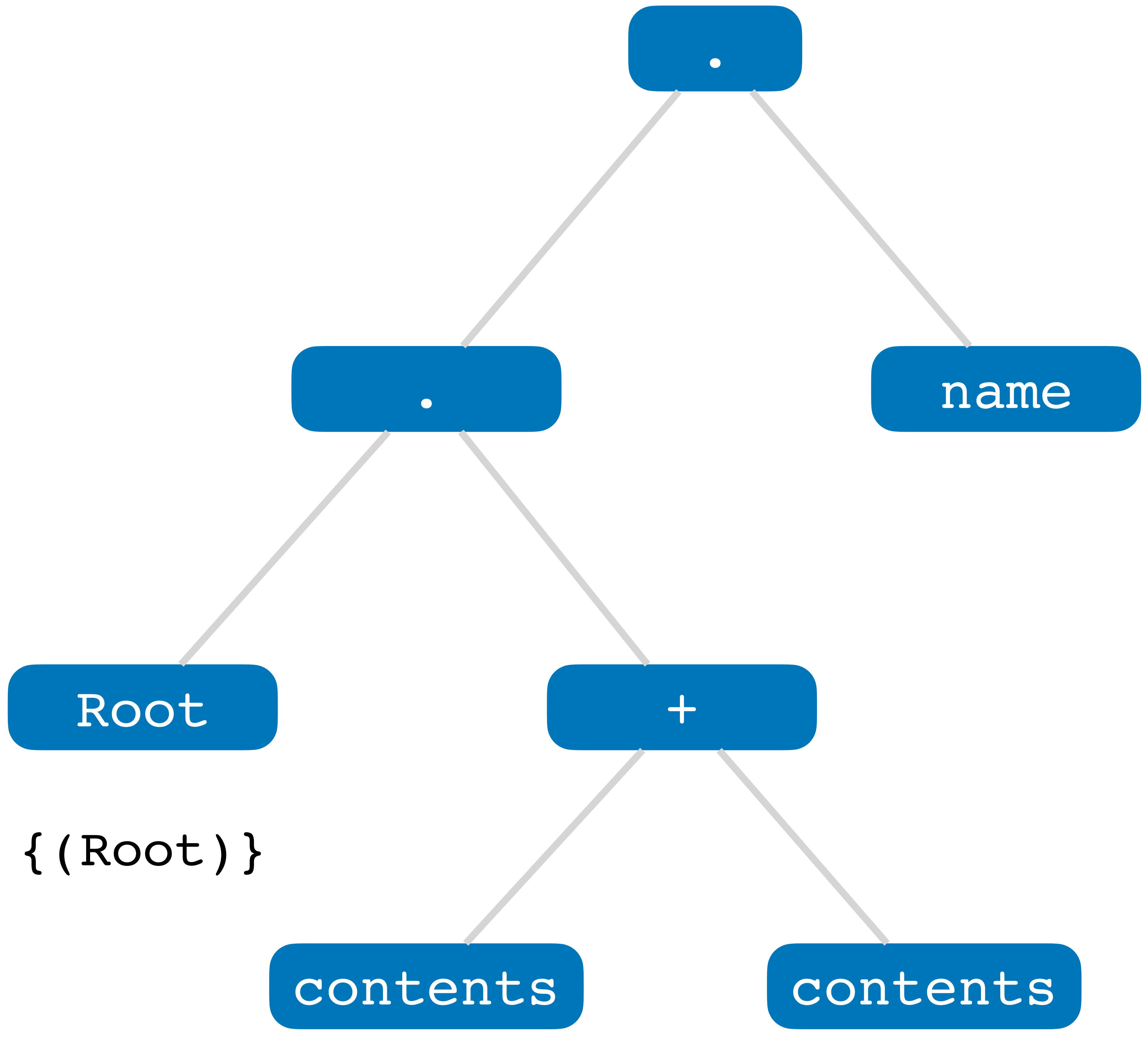
Bounding type inference

`(Root.content).name`



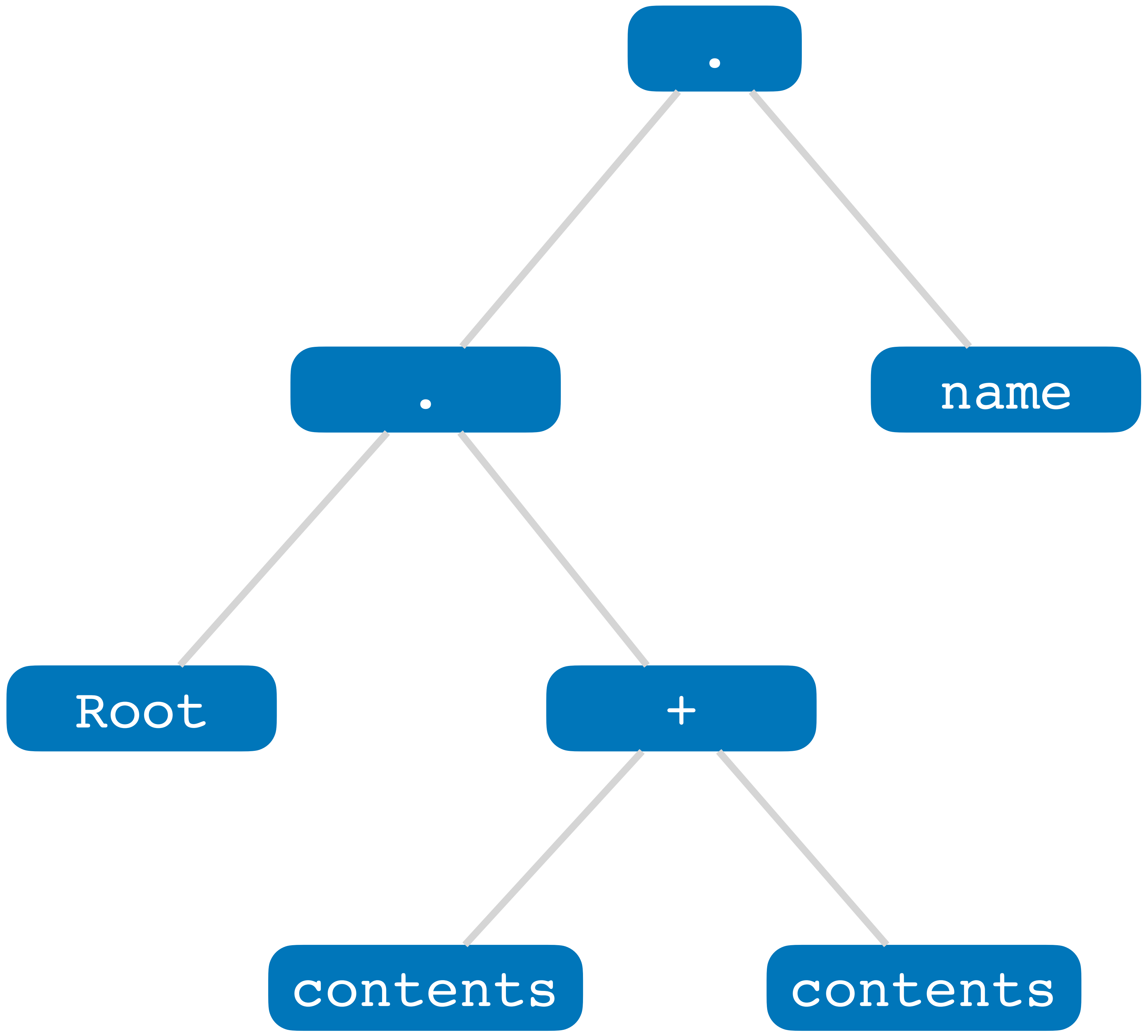
Bounding type inference

`(Root.content).name`



Bounding type inference

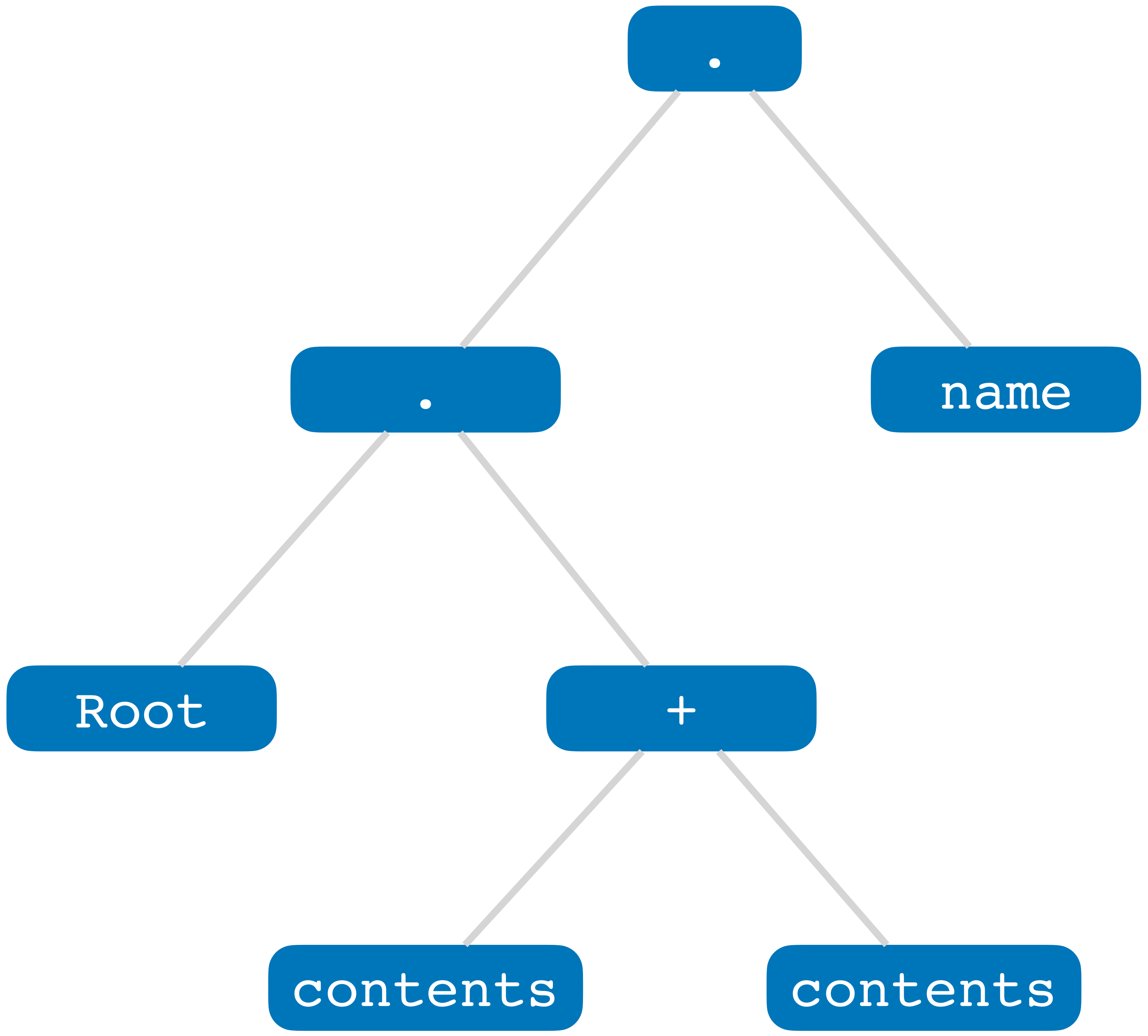
`(Root.content).name`



`{(Root,Entry),($Dir,Entry)}`

Bounding type inference

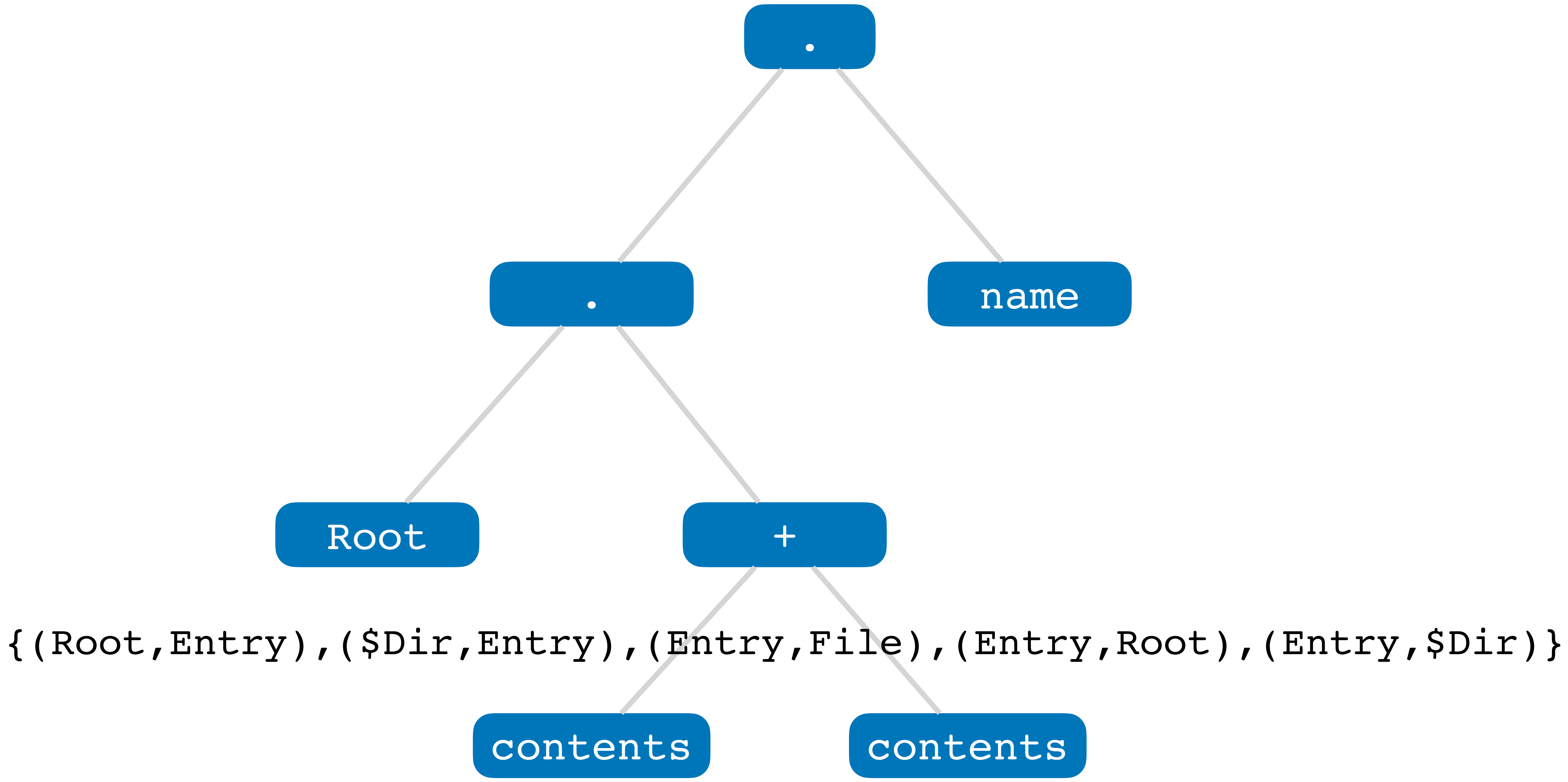
`(Root.content).name`



`{(Entry,File),(Entry,Root),(Entry,$Dir)}`

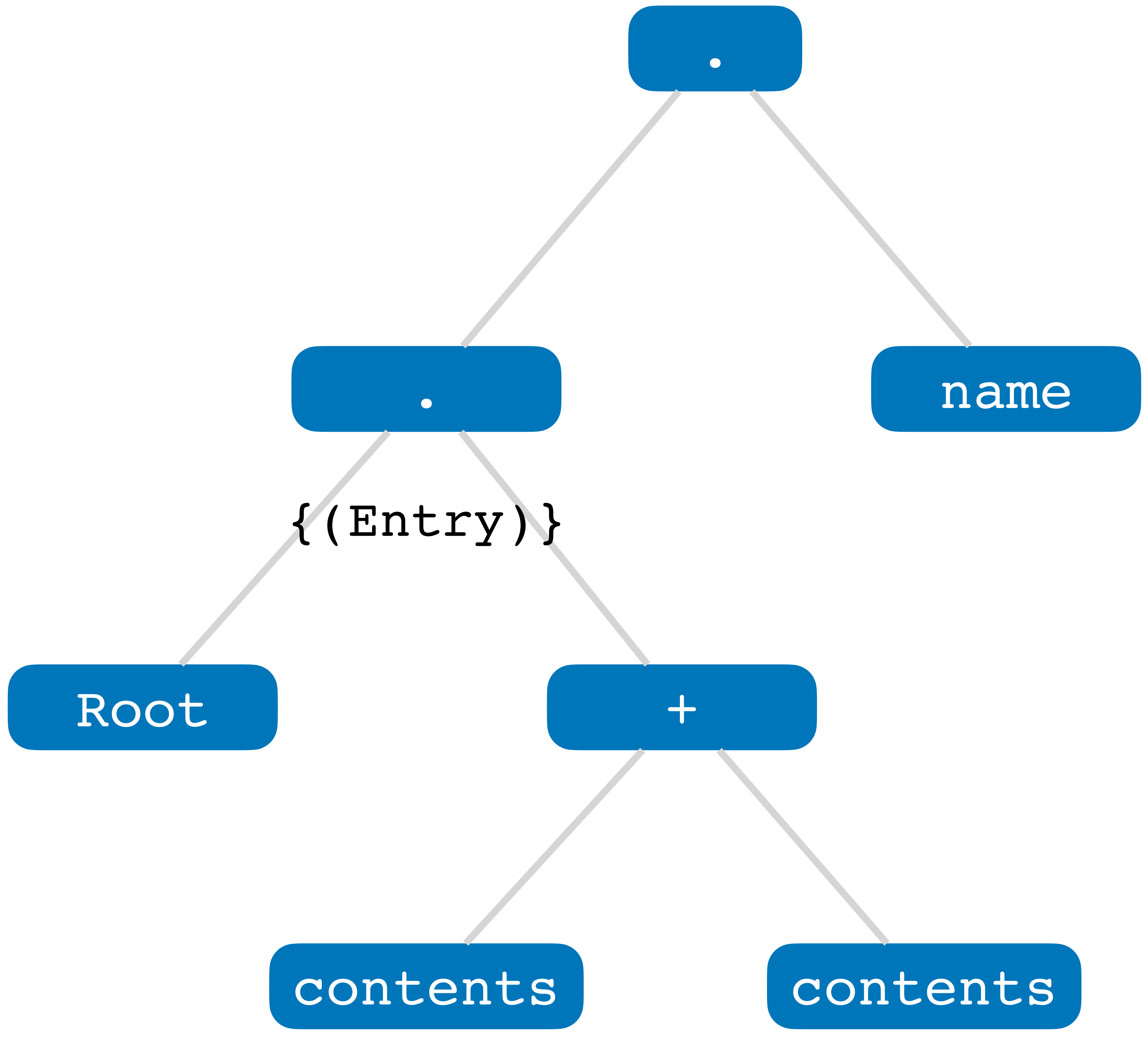
Bounding type inference

(Root.content).name



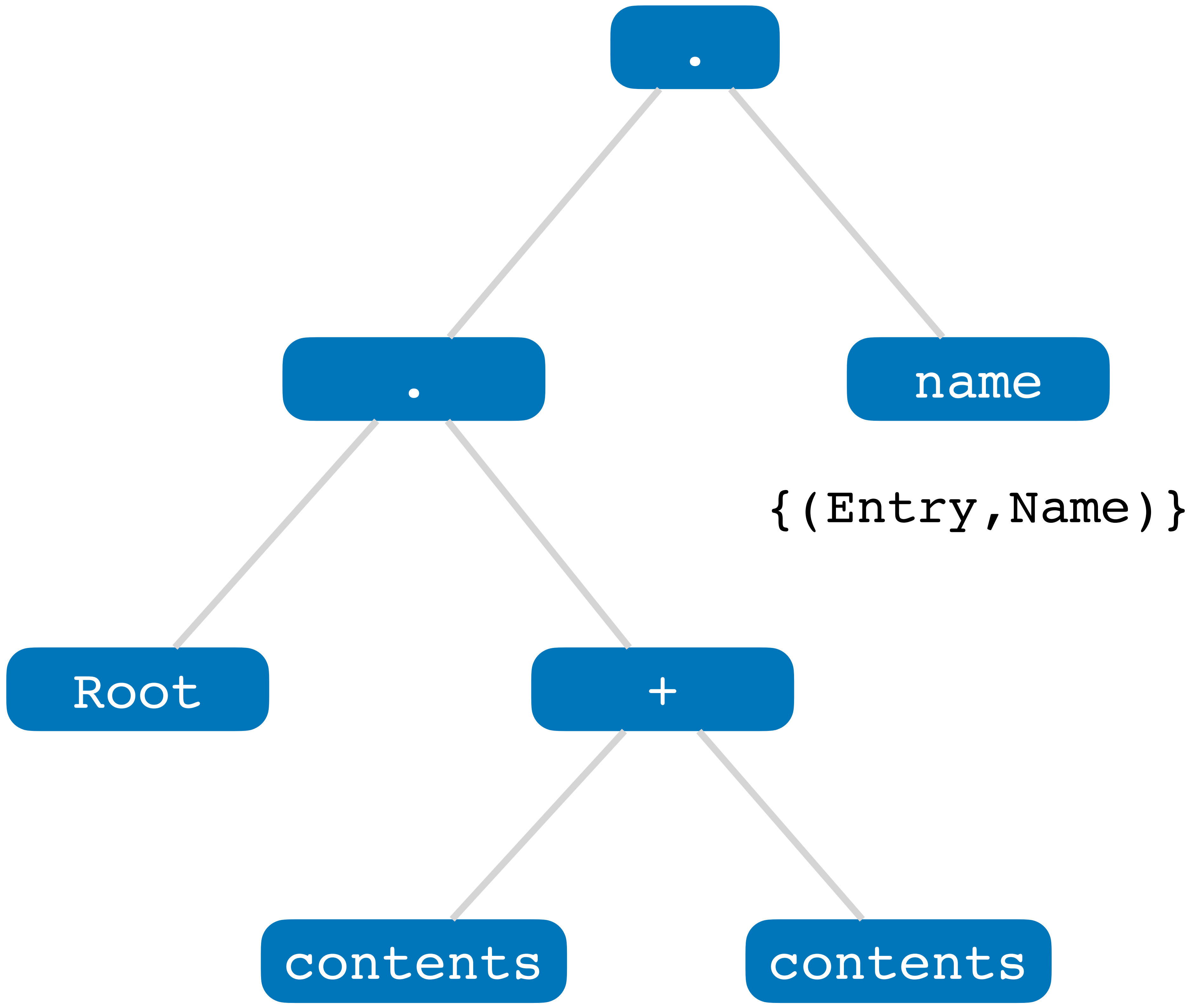
Bounding type inference

`(Root.content).name`



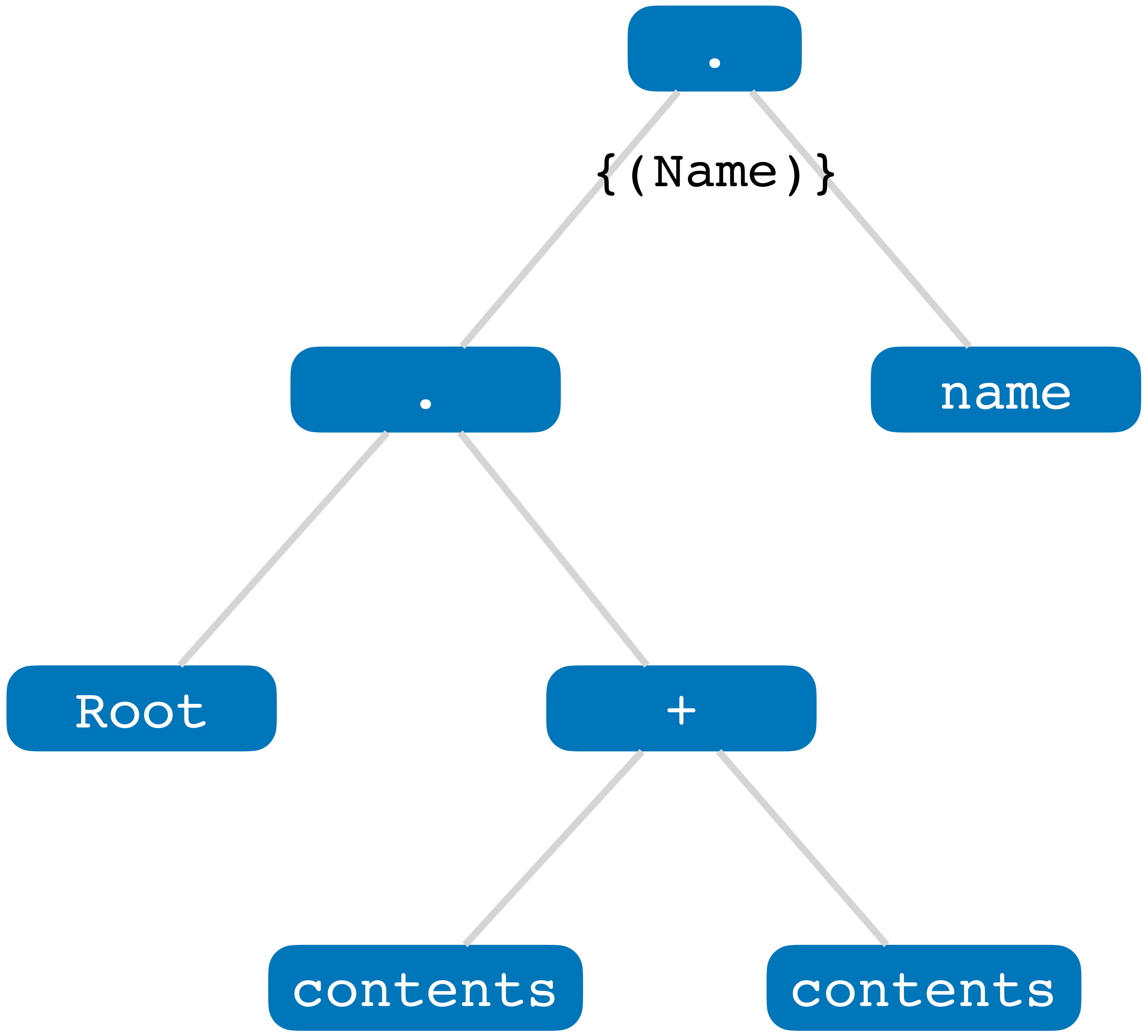
Bounding type inference

`(Root.content).name`



Bounding type inference

`(Root.content).name`

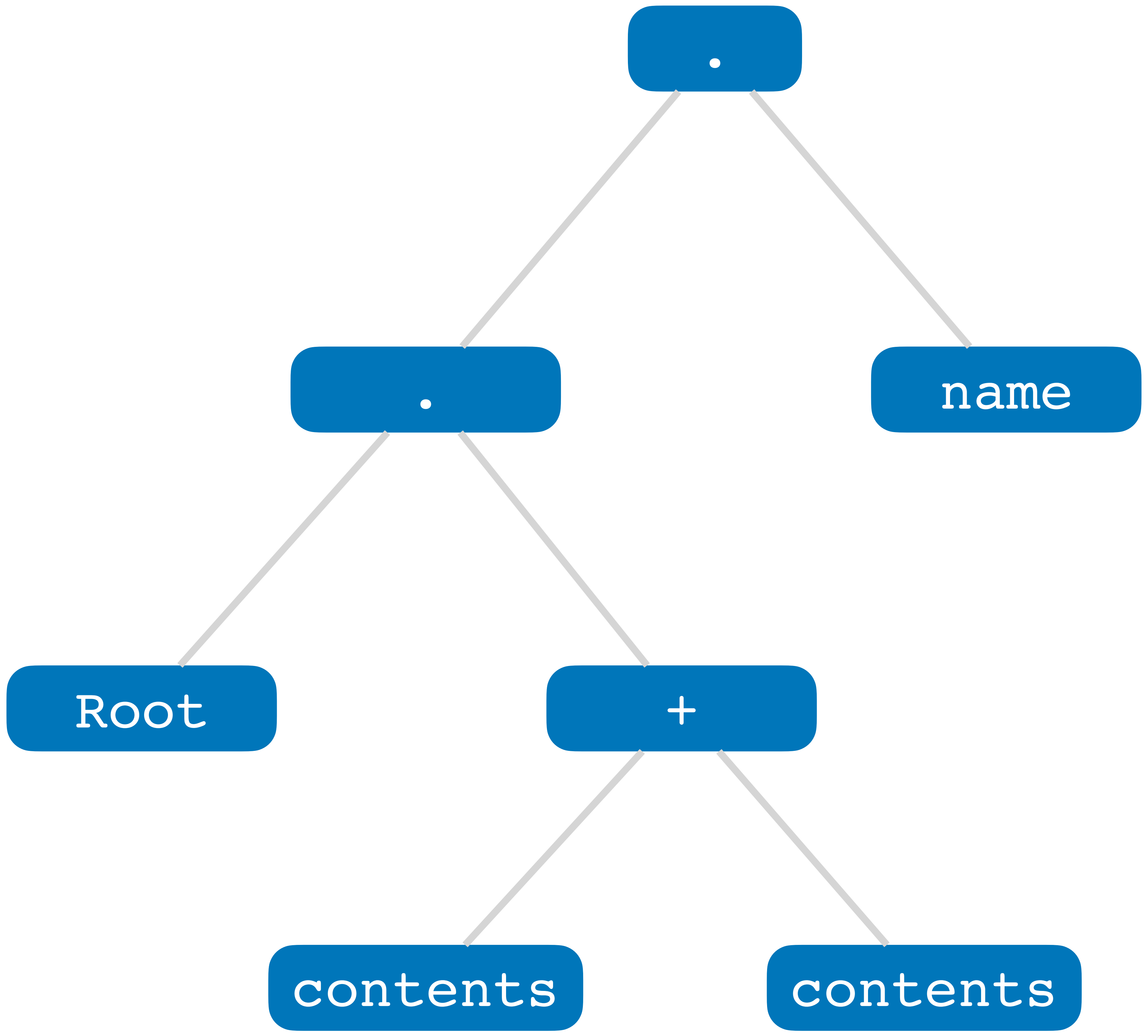


Relevance type inference

- The relevance type of the top expression is equal to its bounding type
- The relevance type of a sub-expression is computed by determining which tuples effectively contributed to the parent expression type

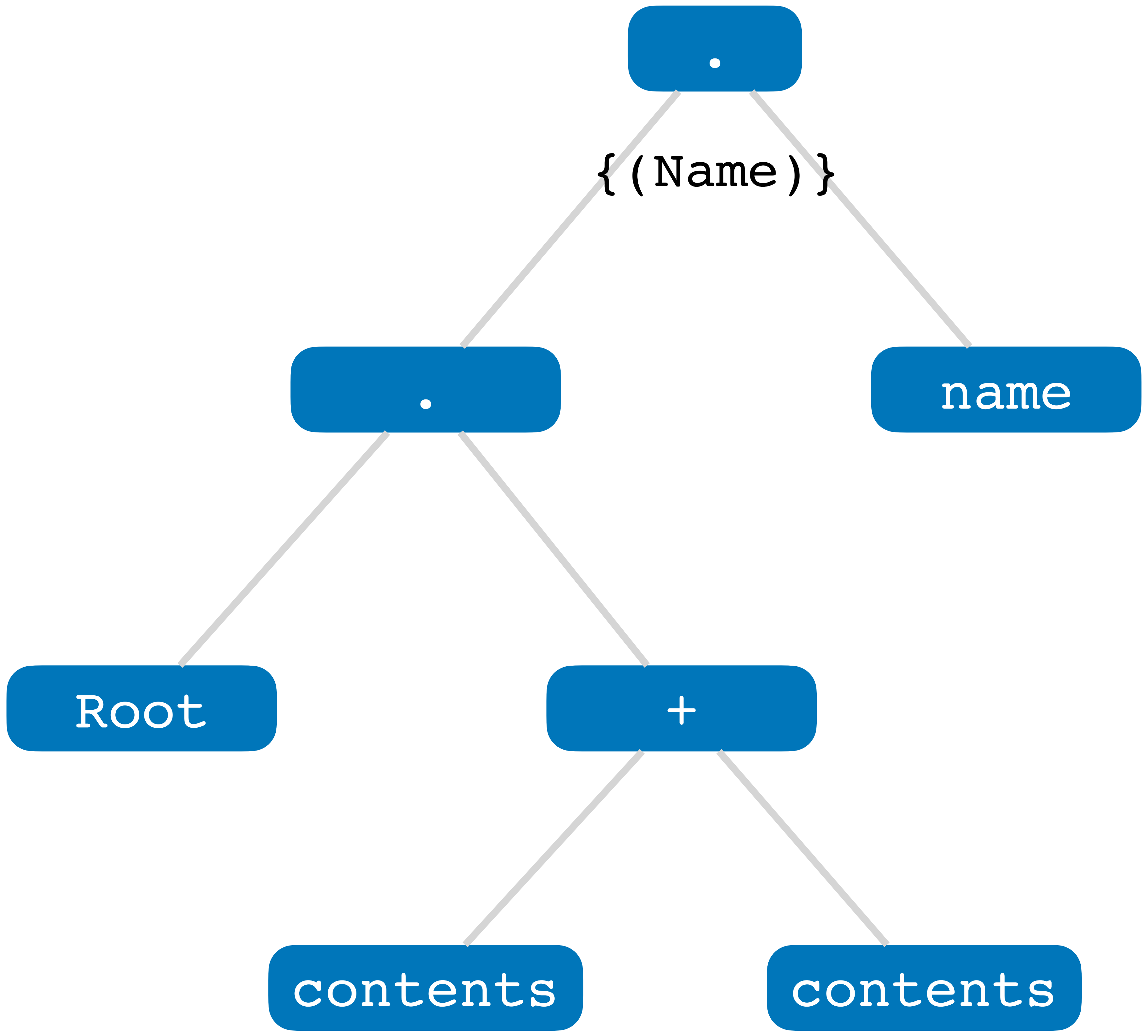
Relevance type inference

```
(Root.content).name
```



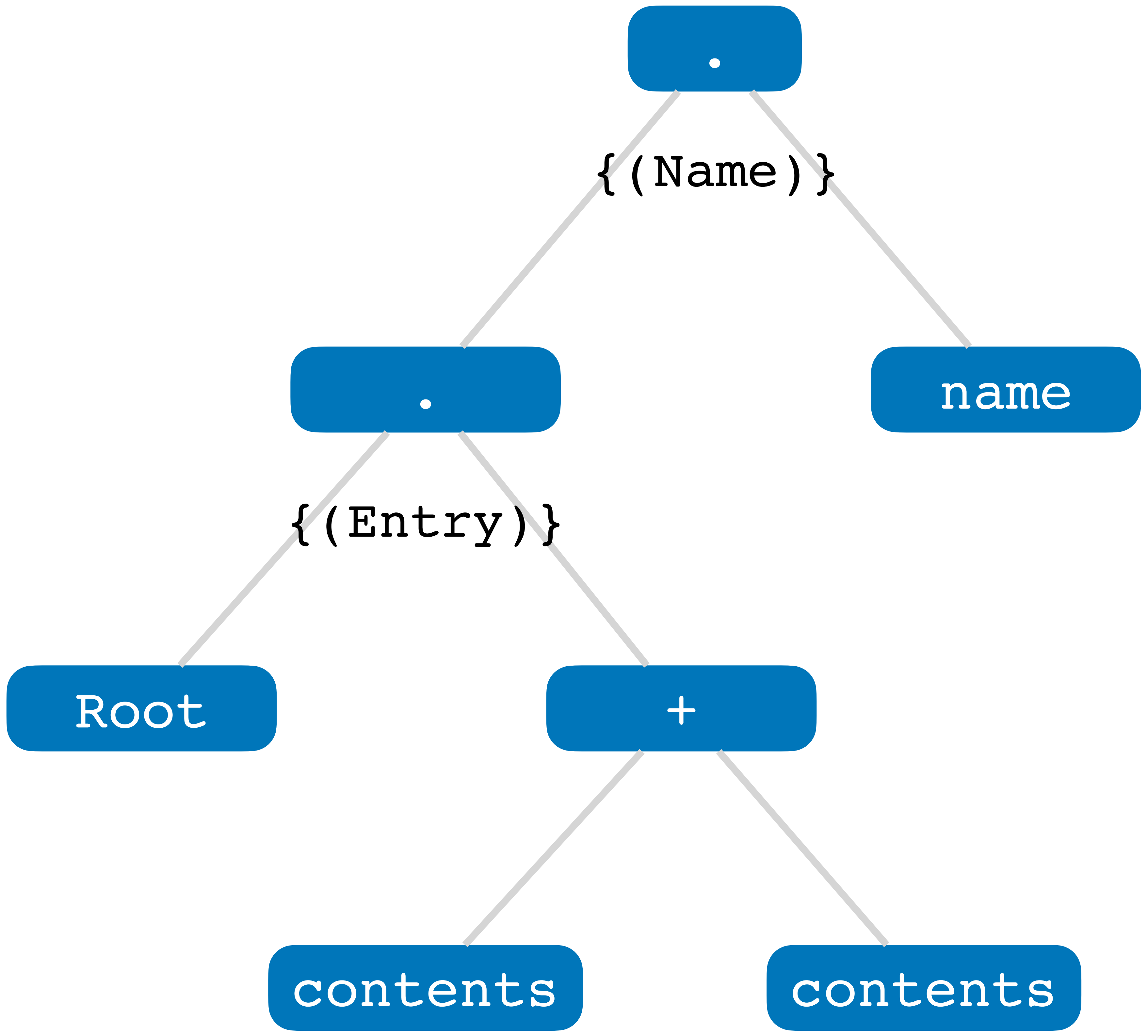
Relevance type inference

`(Root.content).name`



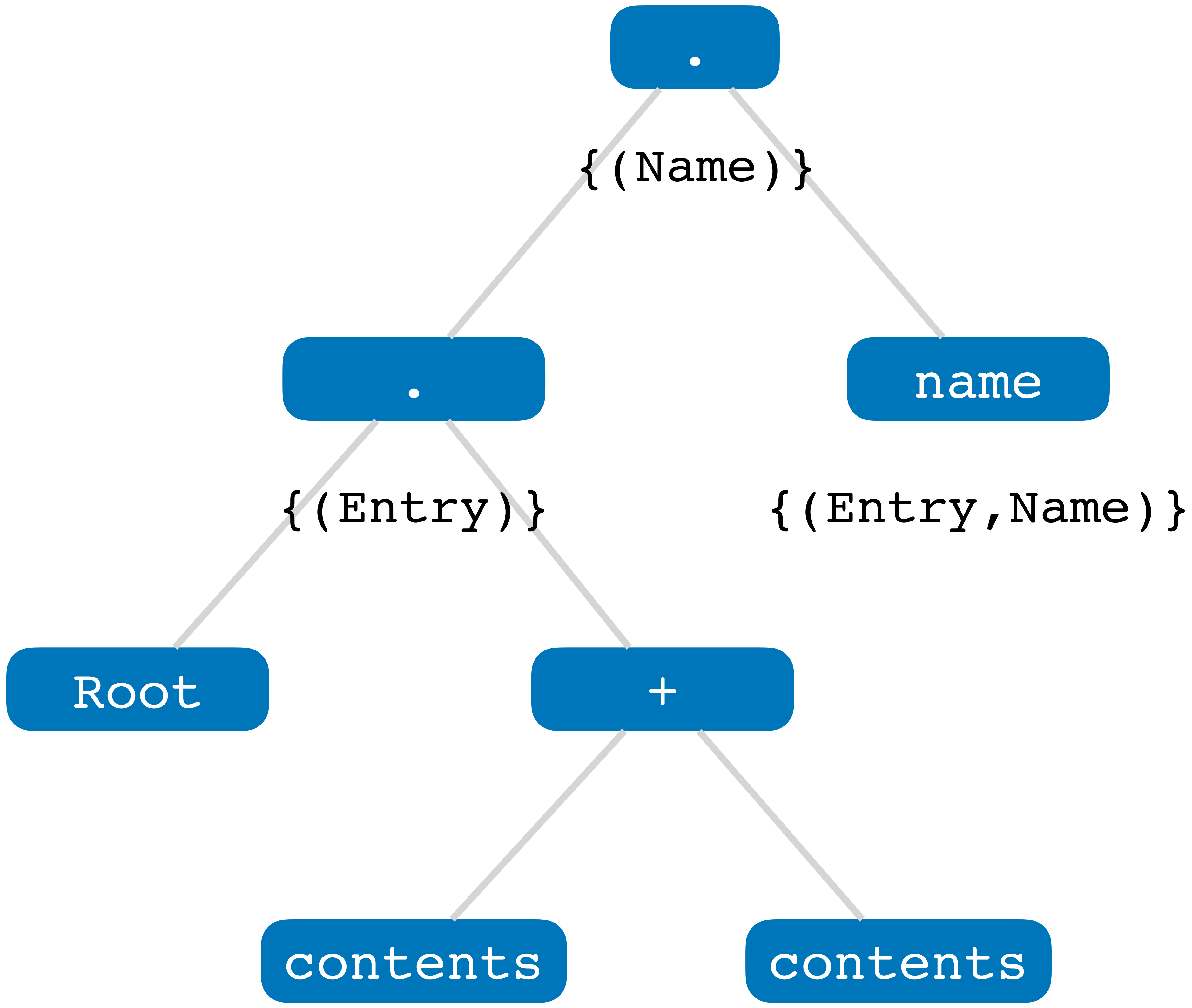
Relevance type inference

`(Root.content).name`



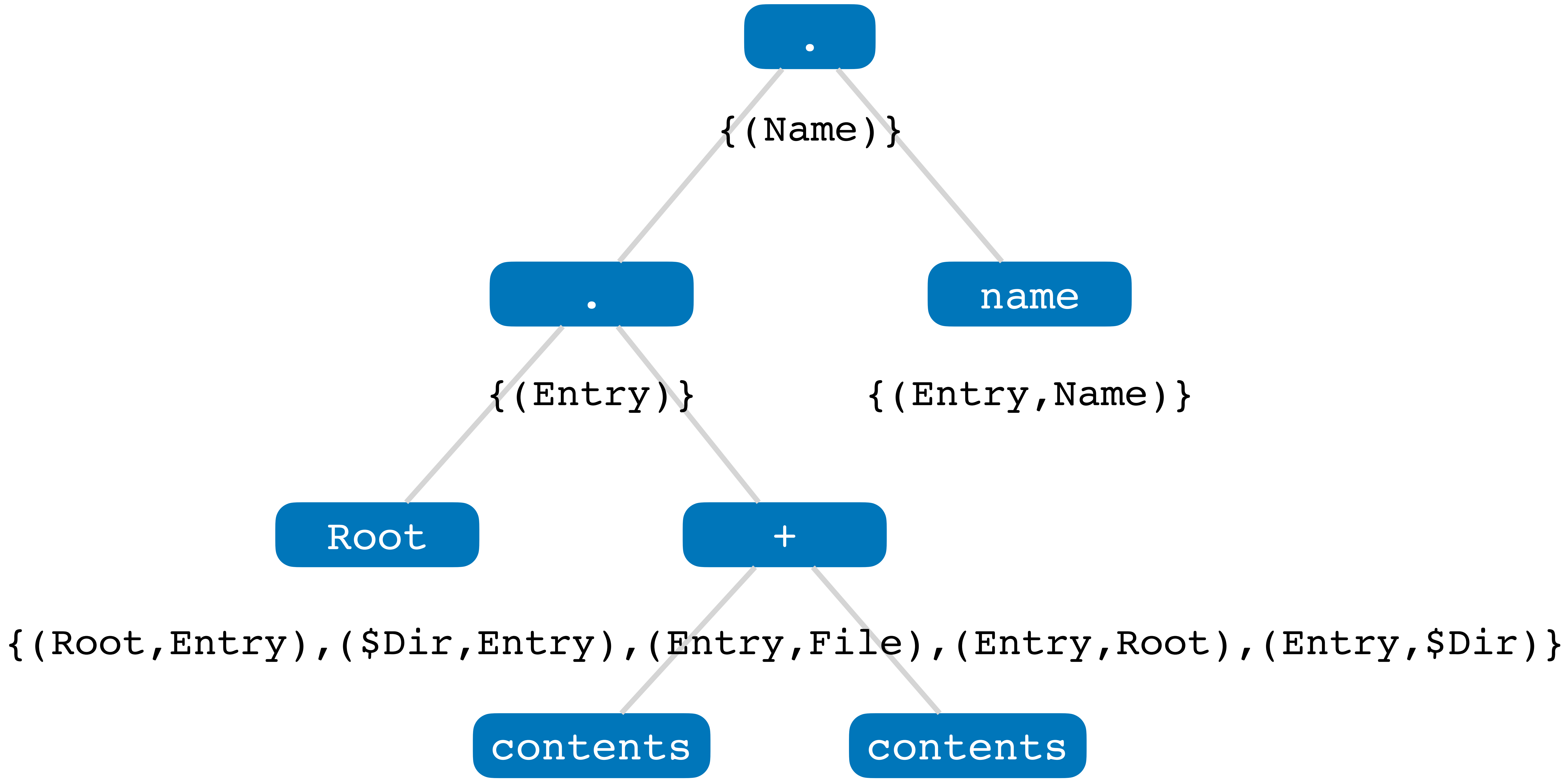
Relevance type inference

`(Root.content).name`



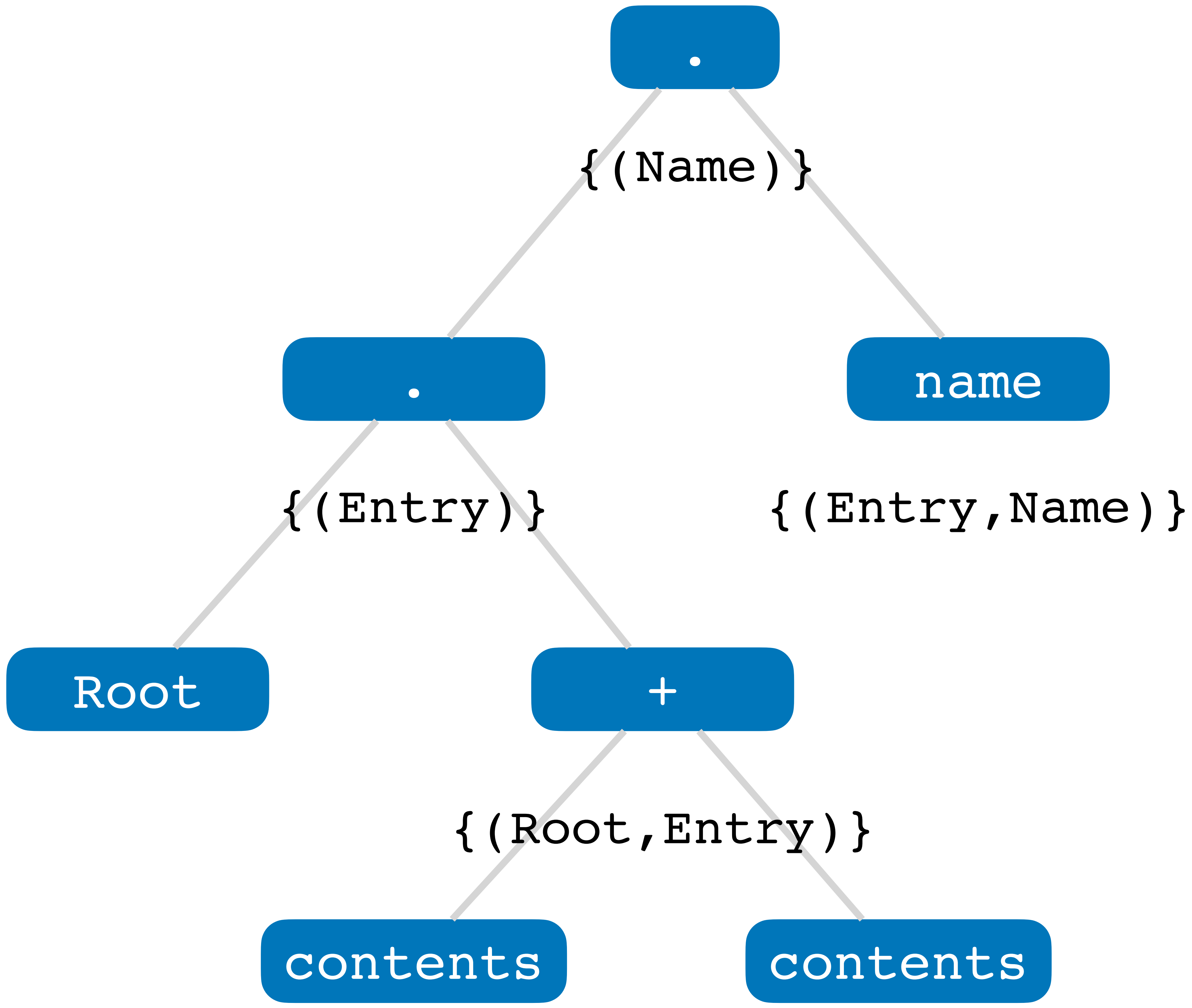
Relevance type inference

`(Root.content).name`



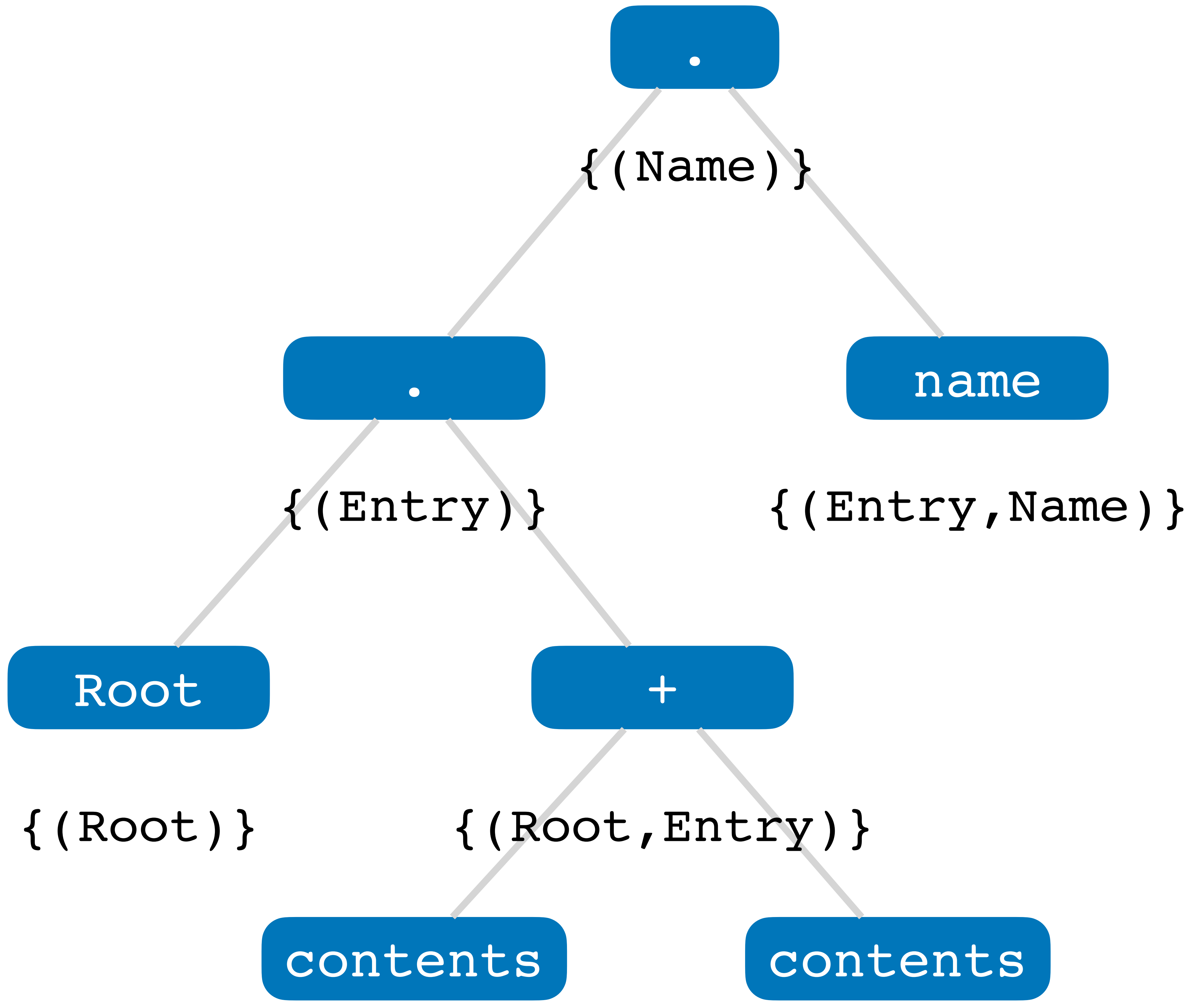
Relevance type inference

`(Root.content).name`



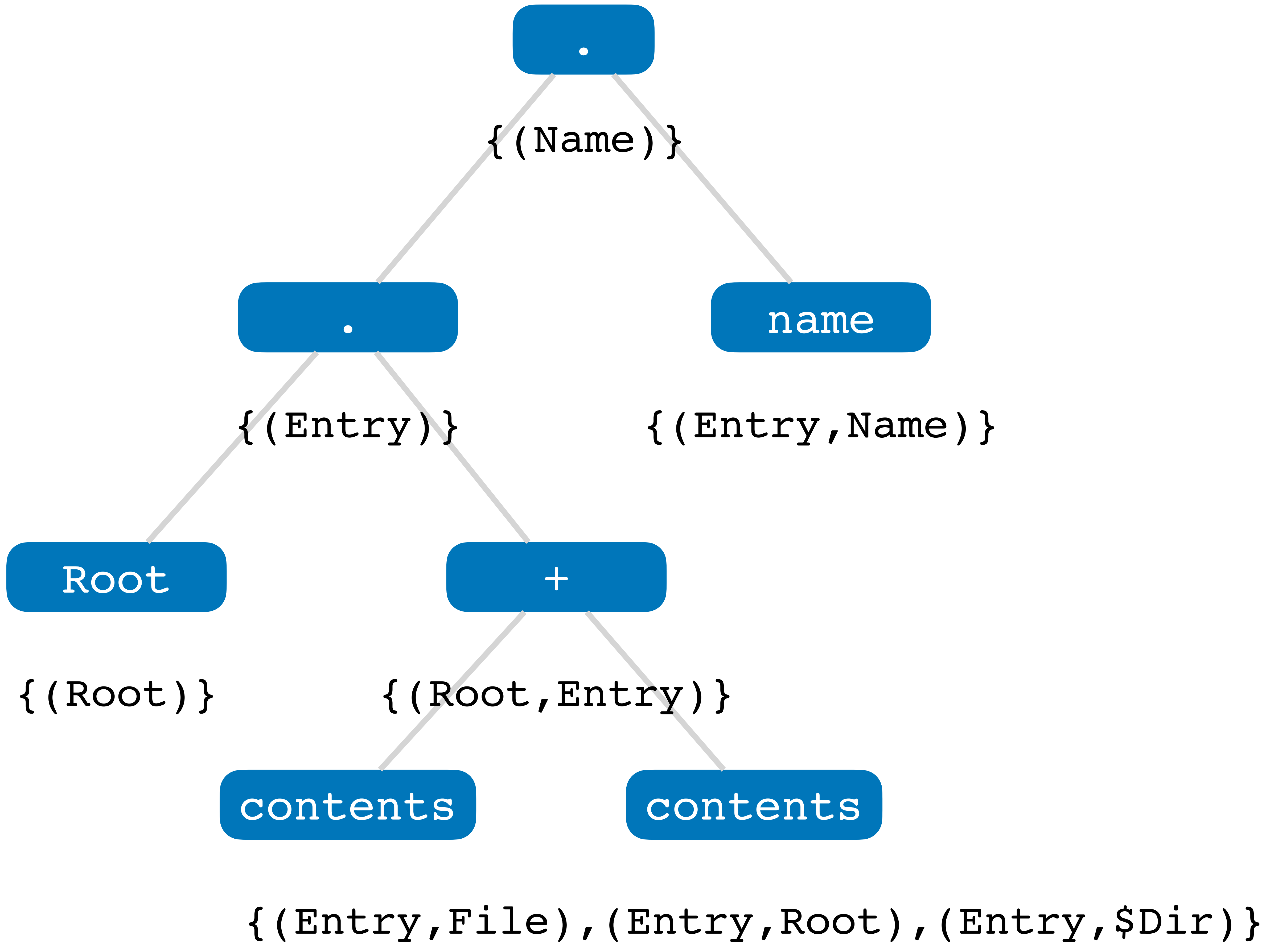
Relevance type inference

(Root.content).name



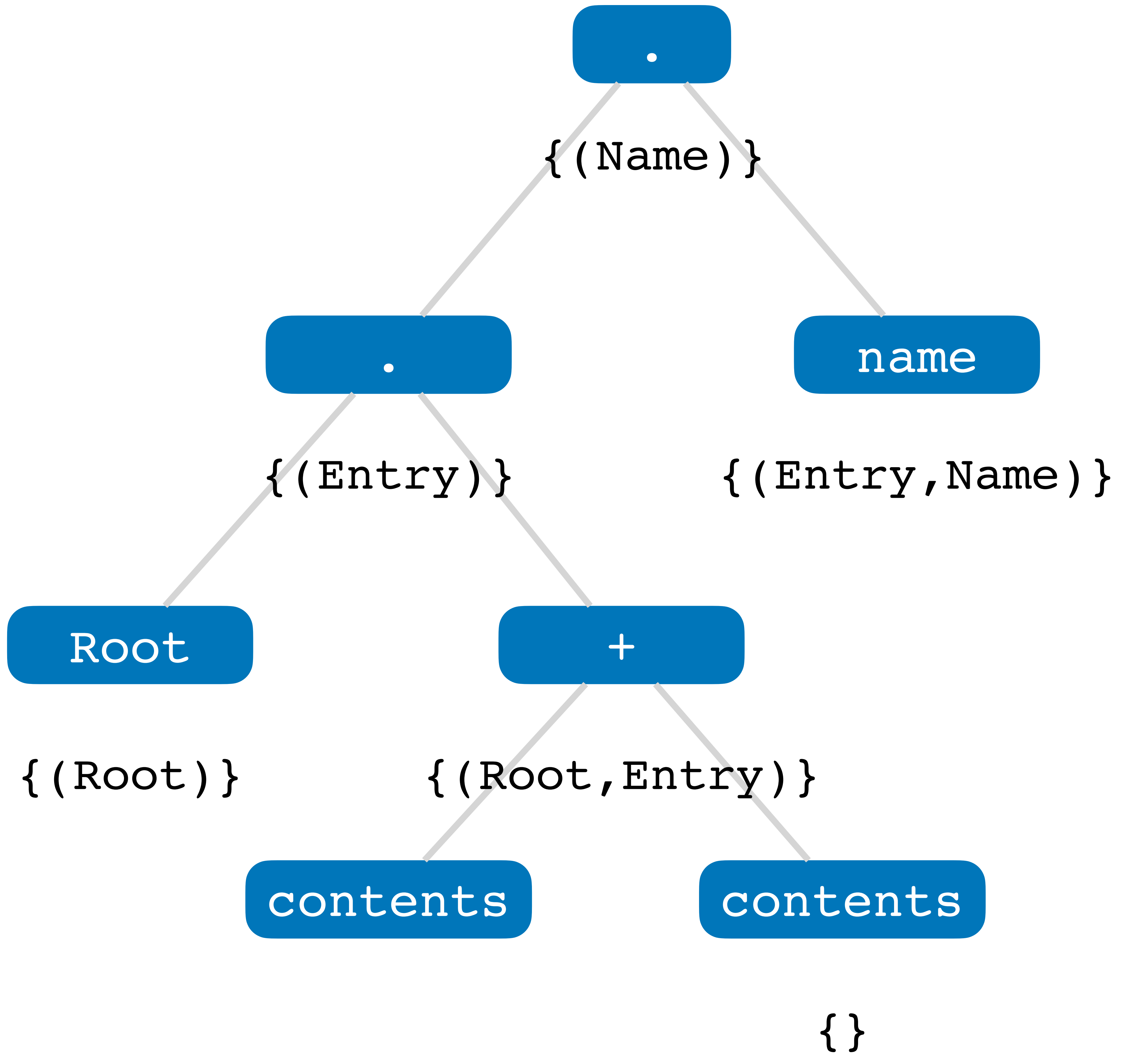
Relevance type inference

`(Root.content).name`



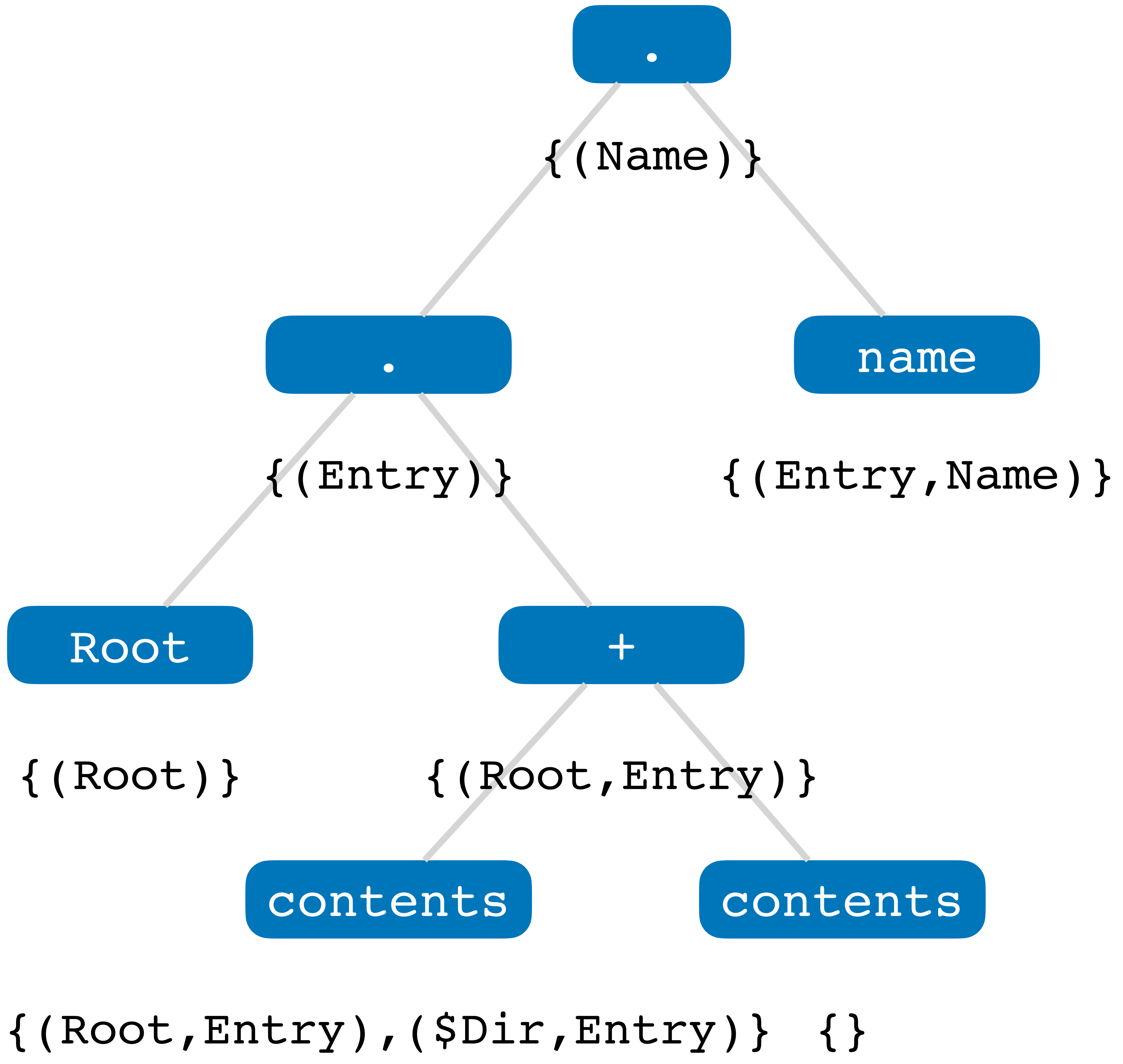
Relevance type inference

```
(Root.content).name
```



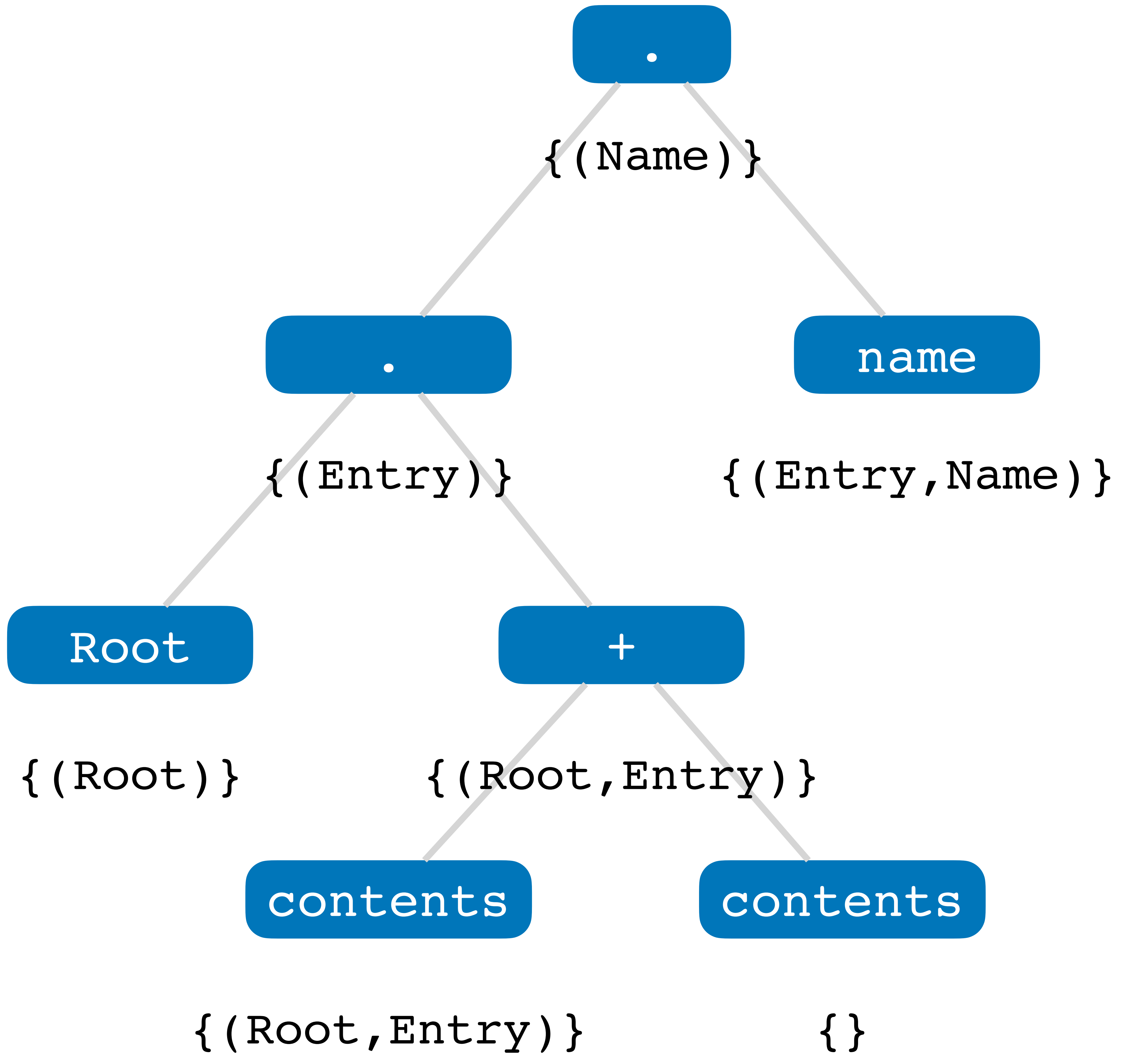
Relevance type inference

`(Root.content).name`



Relevance type inference

`(Root.content).name`



Relational model finding

Architecture

Kodkod

- A Kodkod problem consists of
  - A universe of atoms $\mathscr{U}$
  - A set of relation declarations of shape $r :_a \ r_L \ r_U$
    - a is the *arity* of r
    - r_L is the *lower-bound* of r, tuples that MUST be present in r
    - r_U is the *upper-bound* of r, tuples that MAY be present in r
  - A relational logic formula ϕ

Kodkod

- Kodkod is a *relational model finder*
- It finds a valuation (a model) for the relations such that
  - ϕ is true in that model
  - the valuation of each relation r complies with the *partial-knowledge* declared in the bounds

Alloy ⇄ Kodkod

- Alloy assertions to be checked are negated and conjoined with the facts in the Kodkod formula
  - An assertion is valid if its negation is unsatisfiable
- Alloy fields and atomic signatures are declared in the Kodkod problem
  - Non-atomic signatures are aliased to a disjunction of atomic ones
- Appropriate bounds are inferred from scopes
  - Upper-bounds can be shared between related atomic signatures
    - Further constraints must be added to ensure a sound structural semantics
  - Kodod atom names are meaningless
    - When building an Alloy instance from a Kodkod instance atoms are renamed

Alloy example

```
abstract sig Object {}
sig Dir extends Object {
  entries : set Entry
}
sig File extends Object {}
one sig Root extends Dir {}
sig Entry {}

run { some entries.Entry } for 3 but 2 Entry
```

Kodkod translation

```
{A,B,C,D,E}

$Dir   :1 {} {(A),(B)}
File   :1 {} {(A),(B)}
Root   :1 {(C)} {(C)}
Entry  :1 {} {(D),(E)}
entries :2 {} {(A,D),(A,E),(B,D),(B,E),(C,D),(C,E)}

no File & $Dir
all x : $Dir+Root | x.entries in Entry
entries.univ in $Dir+Root
some entries.Entry
```

Kodkod $\rightleftarrows$ SAT

- A relation r of arity a can be represented by a matrix of boolean variables of size $|\mathcal{U}|^a$

$$r[i_1, \ldots, i_a] = \begin{cases} \top & \text{if } (\mathcal{U}_{i_1}, \ldots, \mathcal{U}_{i_a}) \in r_L \\ r_{i_1, \ldots, i_a} & \text{if } (\mathcal{U}_{i_1}, \ldots, \mathcal{U}_{i_a}) \in r_U \backslash r_L \\ \bot & \text{otherwise} \end{cases}$$

Kodkod ⇄ SAT

$$\texttt{\$Dir} = \begin{bmatrix} d_A \\ d_B \\ \bot \\ \bot \\ \bot \end{bmatrix} \quad \texttt{File} = \begin{bmatrix} f_A \\ f_B \\ \bot \\ \bot \\ \bot \end{bmatrix} \quad \texttt{Root} = \begin{bmatrix} \bot \\ \bot \\ \top \\ \bot \\ \bot \end{bmatrix} \quad \texttt{Entry} = \begin{bmatrix} \bot \\ \bot \\ \bot \\ e_D \\ e_E \end{bmatrix}$$

$$\texttt{entries} = \begin{bmatrix} \bot & \bot & \bot & r_{A,D} & r_{A,E} \\ \bot & \bot & \bot & r_{B,D} & r_{B,E} \\ \bot & \bot & \bot & r_{C,D} & r_{C,E} \\ \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot \end{bmatrix}$$

Kodkod ⇄ SAT

- Relational operators are implemented by matrix operations (in the boolean semiring)

`.`	Multiplication
`+`	Addition
`&`	Hadamard product
…	

- Atomic formulas originate propositional formulas

`in`	Conjunction of point-wise implication
`some`	Disjunction
…	

Kodkod ⇄ SAT

```
some entries.Entry
```

Kodkod ⇄ SAT

some entries.Entry

$$\textbf{some} \begin{bmatrix} \bot & \bot & \bot & r_{A,D} & r_{A,E} \\ \bot & \bot & \bot & r_{B,D} & r_{B,E} \\ \bot & \bot & \bot & r_{C,D} & r_{C,E} \\ \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot \end{bmatrix} \cdot \begin{bmatrix} \bot \\ \bot \\ \bot \\ e_D \\ e_E \end{bmatrix}$$

Kodkod $\rightleftarrows$ SAT

`some` `entries.Entry`

$$\textbf{some} \begin{bmatrix} \bot & \bot & \bot & r_{A,D} & r_{A,E} \\ \bot & \bot & \bot & r_{B,D} & r_{B,E} \\ \bot & \bot & \bot & r_{C,D} & r_{C,E} \\ \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot \end{bmatrix} \cdot \begin{bmatrix} \bot \\ \bot \\ \bot \\ e_D \\ e_E \end{bmatrix}$$

$$\textbf{some} \begin{bmatrix} (r_{A,D} \wedge e_D) \vee (r_{A,E} \wedge e_E) \\ (r_{B,D} \wedge e_D) \vee (r_{B,E} \wedge e_E) \\ (r_{C,D} \wedge e_D) \vee (r_{C,E} \wedge e_E) \\ \bot \\ \bot \end{bmatrix}$$

Kodkod $\rightleftarrows$ SAT

some `entries.Entry`

$$\textbf{some} \begin{bmatrix} \bot & \bot & \bot & r_{A,D} & r_{A,E} \\ \bot & \bot & \bot & r_{B,D} & r_{B,E} \\ \bot & \bot & \bot & r_{C,D} & r_{C,E} \\ \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot \end{bmatrix} \cdot \begin{bmatrix} \bot \\ \bot \\ \bot \\ e_D \\ e_E \end{bmatrix}$$

$$\textbf{some} \begin{bmatrix} (r_{A,D} \land e_D) \lor (r_{A,E} \land e_E) \\ (r_{B,D} \land e_D) \lor (r_{B,E} \land e_E) \\ (r_{C,D} \land e_D) \lor (r_{C,E} \land e_E) \\ \bot \\ \bot \end{bmatrix}$$

$$(r_{A,D} \land e_D) \lor (r_{A,E} \land e_E) \lor (r_{B,D} \land e_D) \lor (r_{B,E} \land e_E) \lor (r_{C,D} \land e_D) \lor (r_{C,E} \land e_E)$$

Quantifiers

- Since the universe is finite quantifiers can be handled by expansion

```
all x : Entry | some entries.x
≡
some entries.{(D)} and some entries.{(E)}
```

- Unfortunately this yields no witnesses to existential quantifiers

```
some x : Entry | some entries.x
≡
some entries.{(D)} or some entries.{(E)}
```

Skolemization

- Skolemization replaces existentially quantified variables by free variables
  - Free variables are implicitly existentially quantified
  - Generates smaller but equisatisfiable formulas
  - Skolemized variables are witnesses that can be shown in the visualiser

some x : Entry | **some** entries.x

≡

$x $:_1$ {} {(D),(E)}

one $x **and** **some** entries.$x

Symmetry breaking

- Kodkod performs several optimisations to decrease SAT complexity
- The most significant is symmetry breaking
  - Since atoms are uninterpreted isomorphic instances are equivalent
  - To avoid returning isomorphic instances a symmetry breaking formula is conjoined to the problem formula
  - For efficiency reasons the technique is not complete
- Besides increasing efficiency symmetry breaking is also useful for validation
  - Otherwise the user would be overwhelmed with isomorphic instances