First-Order Theories & SMT Solvers

Maria João Frade

HASLab - INESC TEC
Departamento de Informática, Universidade do Minho

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Roadmap

- **First-Order Theories**
  - basic concepts; decidability issues;
  - several theories: equality, integers, linear arithmetic, reals, arrays;
  - combining theories;
  - satisfiability modulo theories.
- **SMT solvers**
  - main features;
  - SMT-LIB; SMT's APIs;
  - applications.
- **SMT solvers algoritms** (extra)
  - SMT and SAT solvers integration: "eager" vs "lazy" approach;
  - the basic "lazy offline" approach and its enhancements;
  - DPLL($\mathcal{T}$) framework.

First-Order Theories

Introduction

- When judging the validity of first-order formulas we are typically interested in a particular domain of discourse, which in addition to a specific underlying vocabulary includes also properties that one expects to hold.
- For instance, in formal methods involving the integers, one is not interested in showing that the formula

$$\forall x, y.\ x < y \rightarrow x < y + y$$

is true for all possible interpretations of the symbols $<$ and $+$, but only for those interpretations in which $<$ is the usual ordering over the integers and $+$ is the addition function.

- We are not interested in validity in general but in validity with respect to some *background theory* – a logical theory that fixes the interpretations of certain predicates and function symbols.

Introduction

- Stated differently, we are often interested in moving away from pure logical validity (i.e. validity in all models) towards a more refined notion of validity restricted to a specific class of models.
- There are two ways for specifying such a class of models:
  - To provide a *set of axioms* (sentences that are expected to hold in them).
  - To pinpoint the models of interest.
- *First-order theories* provide a basis for the kind of reasoning just described.

Theories - basic concepts

Let $\mathcal{V}$ be a vocabulary of a first-order language.

- A first-order *theory* $\mathcal{T}$ is a set of $\mathcal{V}$-sentences that is closed under derivability (i.e., $\mathcal{T} \models \phi$ implies $\phi \in \mathcal{T}$).
- A *$\mathcal{T}$-structure* is a $\mathcal{V}$-structure that validates every formula of $\mathcal{T}$.
- A formula ϕ is *$\mathcal{T}$-valid* if every $\mathcal{T}$-structure validates ϕ.
- A formula ϕ is *$\mathcal{T}$-satisfiable* if some $\mathcal{T}$-structure validates ϕ.
- Two formulae ϕ and ψ are *$\mathcal{T}$-equivalent* if $\mathcal{T} \models \phi \leftrightarrow \psi$ (i.e, for every $\mathcal{T}$-structure $\mathcal{M}$, $\mathcal{M} \models \phi$ iff $\mathcal{M} \models \psi$).

Theories - basic concepts

Let $\mathcal{T}$ be a first-order theory.

- $\mathcal{T}$ is said to be a *consistent* theory if at least one $\mathcal{T}$-structure exists.
- $\mathcal{T}$ is said to be a *complete* theory if, for every $\mathcal{V}$-sentence ϕ, either $\mathcal{T} \models \phi$ or $\mathcal{T} \models \neg\phi$.
- $\mathcal{T}$ is said to be a *decidable* theory if there exists a decision procedure for checking $\mathcal{T}$-validity.

Theories - basic concepts

- Let K be a class of $\mathcal{V}$-structures. The *theory of K*, denoted by $\mathsf{Th}(K)$, is the set of sentences valid in all members of K, i.e., $\mathsf{Th}(K) = \{\phi \mid \mathcal{M} \models \phi, \text{for all } \mathcal{M} \in K\}$.
- Given a set of $\mathcal{V}$-sentences Γ, the class of *models for Γ*, denoted by $\mathsf{Mod}(\Gamma)$, is defined as $\mathsf{Mod}(\Gamma) = \{\mathcal{M} \mid \text{for all } \phi \in \Gamma, \mathcal{M} \models \phi\}$.
- A subset $\mathcal{A} \subseteq \mathcal{T}$ is called an *axiom set* for the theory $\mathcal{T}$, when $\mathcal{T}$ is the deductive closure of $\mathcal{A}$, i.e. $\phi \in \mathcal{T}$ iff $\mathcal{A} \models \phi$.
- A theory $\mathcal{T}$ is *finitely* (resp. *recursively*) *axiomatizable* if it possesses a finite (resp. recursive) set of axioms.
- A *fragment* of a theory is a syntactically-restricted subset of formulae of the theory.

Theories - some results

- For a given $\mathcal{V}$-structure $\mathcal{M}$, the theory $\mathsf{Th}(\mathcal{M})$ (of a single-element class of $\mathcal{V}$-structures) is complete.
  - These semantically defined theories are useful when one is interested in reasoning in some specific mathematical domain such as the natural numbers, rational numbers, etc.
  - Such theories may lack an axiomatisation, which seriously compromises its use in purely deductive reasoning.
- If a theory is complete and recursive axiomatizable, it can be shown to be decidable.

Theories - decidability problem

- The decidability criterion for $\mathcal{T}$-validity is crucial for mechanised reasoning in the theory $\mathcal{T}$.
- It may be necessary (or convenient) to restrict the class of formulas under consideration to a suitable *fragment* (i.e., syntactical constraint).
- The $\mathcal{T}$-validity problem in a fragment refers to the decision about whether or not $\phi \in \mathcal{T}$ when ϕ belongs to the fragment under consideration.
- A fragment of interest is the *quantifier-free (QF) fragment*.

Equality and uninterpreted functions $\mathcal{T}_\mathsf{E}$

- The **vocabulary** of the theory of *equality* $\mathcal{T}_\mathsf{E}$ consists of
  - equality ($=$), which is the only interpreted symbol (whose meaning is defined via the axioms of $\mathcal{T}_\mathsf{E}$);
  - constant, function and predicate symbols, which are uninterpreted (except as they relate to $=$).
- **Axioms:**
  - *reflexivity:* $\forall x.\ x = x$
  - *symmetry:* $\forall x, y.\ x = y \rightarrow y = x$
  - *transitivity:* $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$
  - *congruence for functions:* for every function $f \in \mathcal{T}$ with $\mathsf{ar}(f) = n$,

    $$\forall \overline{x}, \overline{y}.\ (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$$

  - *congruence for predicates:* for every predicate $P \in \mathcal{T}$ with $\mathsf{ar}(P) = n$,

    $$\forall \overline{x}, \overline{y}.\ (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow (P(x_1, \ldots, x_n) \leftrightarrow P(y_1, \ldots, y_n))$$

- $\mathcal{T}_\mathsf{E}$-validity is undecidable, but efficiently decidable for the QF fragment.

Peano arithmetic $\mathcal{T}_{\mathsf{PA}}$

- The theory of *Peano arithmetic* $\mathcal{T}_{\mathsf{PA}}$ (1889) is a first-order approximation of the theory of natural numbers.
- **Vocabulary:** $\mathcal{V}_{\mathsf{PA}} = \{0, 1, +, \times, =\}$
- **Axioms:**
  - axioms of $\mathcal{T}_\mathsf{E}$
  - $\forall x.\ \neg(x + 1 = 0)$ *(zero)*
  - $\forall x, y.\ x + 1 = y + 1 \rightarrow x = y$ *(successor)*
  - $\forall x.\ x + 0 = x$ *(plus zero)*
  - $\forall x, y.\ x + (y + 1) = (x + y) + 1$ *(plus successor)*
  - $\forall x.\ x \times 0 = 0$ *(time zero)*
  - $\forall x, y.\ x \times (y + 1) = (x \times y) + x$ *(times successor)*
  - for every formula ϕ with $\mathsf{FV}(\phi) = \{x\}$ *(axiom schema of induction)*

    $$\phi[0/x] \wedge (\forall x.\ \phi \rightarrow \phi[x + 1/x]) \rightarrow \forall x.\ \phi$$

- $\mathcal{T}_{\mathsf{PA}}$ is incomplete and undecidable, even for the quantifier-free fragment.

Peano arithmetic $\mathcal{T}_{\mathsf{PA}}$

- The incompleteness result is indeed striking because, at the end of the 19th century, G. Peano had given a set of axioms that were shown to characterise natural numbers up to isomorphism. One of these axioms – the *axiom of induction* – involves quantification over arbitrary properties of natural numbers: "*for every unary predicate P, if $P(0)$ and $\forall n.\ P(n) \rightarrow P(n+1)$ then $\forall n.\ P(n)$*", which is not a first-order axiom.

- It is however important to notice that the approximation done by a first-order axiom scheme that replaces the arbitrary property P by a first-order formula ϕ with a free variable x:

$$\phi[0/x] \wedge (\forall x.\ \phi \rightarrow \phi[x+1/x]) \rightarrow \forall x.\ \phi$$

restrict reasoning to properties that are definable by first-order formulas, which can only capture a small fragment of all possible properties of natural number. (Recall that the set of first-order formulas is countable while the set of arbitrary properties of natural numbers is $\mathcal{P}(\mathbb{N})$, which is uncountable.)

Presburger arithmetic $\mathcal{T}_{\mathbb{N}}$

- The theory of *Presburger arithmetic* $\mathcal{T}_{\mathbb{N}}$ is the additive fragment of the theory of Peano.
- **Vocabulary:** $\mathcal{V}_{\mathbb{N}} = \{0, 1, +, =\}$
- **Axioms:**
  - axioms of $\mathcal{T}_{\mathsf{E}}$
  - $\forall x.\ \neg(x+1=0)$ *(zero)*
  - $\forall x, y.\ x+1 = y+1 \rightarrow x = y$ *(successor)*
  - $\forall x.\ x+0 = x$ *(plus zero)*
  - $\forall x, y.\ x+(y+1) = (x+y)+1$ *(plus successor)*
  - for every formula ϕ with $\mathsf{FV}(\phi) = \{x\}$ *(axiom schema of induction)*

$$\phi[0/x] \wedge (\forall x.\ \phi \rightarrow \phi[x+1/x]) \rightarrow \forall x.\ \phi$$

- $\mathcal{T}_{\mathbb{N}}$ is both complete and decidable (Presburger, 1929), but it has double exponential complexity.

Linear integer arithmetic $\mathcal{T}_{\mathbb{Z}}$

- **Vocabulary:** $\mathcal{V}_{\mathbb{Z}} = \{\ldots, -2, -1, 0, 1, 2, \ldots, -3\cdot, -2\cdot, 2\cdot, 3\cdot, \ldots, +, -, >, =\}$
- Each symbol is interpreted with its standard mathematical meaning in $\mathbb{Z}$.
  - Note: $\ldots, -3\cdot, -2\cdot, 2\cdot, 3\cdot, \ldots$ are unary functions. For example, the intended meaning of $3 \cdot x$ is $x+x+x$, and of $-2 \cdot x$ is $-x-x$.

$\mathcal{T}_{\mathbb{Z}}$ and $\mathcal{T}_{\mathbb{N}}$ have the same expressiveness

- For every formula of $\mathcal{T}_{\mathbb{Z}}$ there is an equisatisfiable formula of $\mathcal{T}_{\mathbb{N}}$.
- For every formula of $\mathcal{T}_{\mathbb{N}}$ there is an equisatisfiable formula of $\mathcal{T}_{\mathbb{Z}}$.

Let ϕ be a formula of $\mathcal{T}_{\mathbb{Z}}$ and ψ a formula of $\mathcal{T}_{\mathbb{N}}$. ϕ and ψ are *equisatisfiable* if

$$\phi \text{ is } \mathcal{T}_{\mathbb{Z}}\text{-satisfiable} \quad \text{iff} \quad \psi \text{ is } \mathcal{T}_{\mathbb{N}}\text{-satisfiable}$$

- $\mathcal{T}_{\mathbb{Z}}$ is both complete and decidable via the rewriting of $\mathcal{T}_{\mathbb{Z}}$-formulae into $\mathcal{T}_{\mathbb{N}}$-formulae.

Linear rational arithmetic $\mathcal{T}_{\mathbb{Q}}$

- The full theory of rational numbers (with addition and multiplication) is *undecidable*, since the property of being a natural number can be encoded in it.
- But the theory of *linear arithmetic over rational numbers* $\mathcal{T}_{\mathbb{Q}}$ is decidable, and actually more efficiently than the corresponding theory of integers.
- **Vocabulary:** $\mathcal{V}_{\mathbb{Q}} = \{0, 1, +, -, =, \geq\}$
- **Axioms:** 10 axioms (see Manna's book)
- Rational coefficients can be expressed in $\mathcal{T}_{\mathbb{Q}}$.

The formula $\frac{5}{2}x + \frac{4}{3}y \leq 6$ can be written as the $\mathcal{T}_{\mathbb{Q}}$-formula

$$36 \geq 15x + 8y$$

- $\mathcal{T}_{\mathbb{Q}}$ is decidable and its quantifier-free fragment is efficiently decidable.

Reals $\mathcal{T}_{\mathbb{R}}$

- Surprisingly, the *theory of reals* $\mathcal{T}_{\mathbb{R}}$ is decidable even in the presence of multiplication and quantifiers.
- **Vocabulary:** $\mathcal{V}_{\mathbb{R}} = \{0, 1, +, \times, -, =, \geq\}$
- **Axioms:** 17 axioms (see Manna's book)

The inclusion of multiplication allows a formula like $\exists x.\ x^2 = 3$ to be expressed (x^2 abbreviates $x \times x$). This formula should be $\mathcal{T}_{\mathbb{R}}$-valid, since the assignment $x \mapsto \sqrt{3}$ satisfies $x^2 = 3$.

- $\mathcal{T}_{\mathbb{R}}$ is decidable (Tarski, 1949). However, it has a high time complexity (doubly exponential).

Difference arithmetic

- *Difference logic* is a fragment (a sub-theory) of linear arithmetic.
- Atomic formulas have the form $x - y \leq c$, for variables x and y and constant c.
- Conjunctions of difference arithmetic inequalities can be checked very efficiently for satisfiability by searching for negative cycles in weighted directed graphs.

  Graph representation: each variable corresponds to a node, and an inequality of the form $x - y \leq c$ corresponds to an edge from y to x with weight c.
- The quantifier-free satisfiability problem is solvable in $\mathcal{O}(|V||E|)$.

Arrays $\mathcal{T}_{\mathsf{A}}$ and $\mathcal{T}_{\mathsf{A}}^{=}$

- Arrays are modeled in logic as applicative data structures.
- **Vocabulary:** $\mathcal{V}_{\mathsf{A}} = \{read, write, =\}$
- **Axioms:**
  - (reflexivity), (symmetry) and (transitivity) of $\mathcal{T}_{\mathsf{E}}$
  - $\forall a, i, j.\ i = j \rightarrow read(a, i) = read(a, j)$
  - $\forall a, i, j, v.\ i = j \rightarrow read(write(a, i, v), j) = v$
  - $\forall a, i, j, v.\ \neg(i = j) \rightarrow read(write(a, i, v), j) = read(a, j)$
- $=$ is only defined for array elements.
- $\mathcal{T}_{\mathsf{A}}^{=}$ is the theory $\mathcal{T}_{\mathsf{A}}$ plus an axiom (extensionality) to capture $=$ on arrays.
  - $\forall a, b.\ (\forall i.\ read(a, i) = read(b, i)) \leftrightarrow a = b$
- Both $\mathcal{T}_{\mathsf{A}}$ and $\mathcal{T}_{\mathsf{A}}^{=}$ are undecidable. But their quantifier-free fragments are decidable.
- Alternative fragments are often preferred that subsume the quantifier-free fragment (allowing restricted forms of index quantification).

Other theories

- *Fixed-size bit-vectors*
  - Model bit-level operations of machine words, including 2^n-modular operations (where n is the word size), shift operations, etc.
  - Decision procedures for the theory of fixed-size bit vectors often rely on appropriate encodings in propositional logic.
- *Algebraic data structures*
  - The theories describe data structures that are ubiquitous in programming like lists, stacks, binary trees, etc.
  - These theories are built around the theory of equality with uninterpreted functions, and are normally efficiently decidable for the quantifier-free fragment.

Combining theories

- In practice, the most of the formulae we want to check need a combination of theories.

Checking $x + 2 = y \to f(read(write(a, x, 3), y - 2)) = f(y - x + 1)$
involves 3 theories: equality and uninterpreted functions, arrays and arithmetic.

- Given theories $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \{=\}$, the *combined theory* $\mathcal{T}_1 \cup \mathcal{T}_2$ has vocabulary $\mathcal{V}_1 \cup \mathcal{V}_2$ and axioms $A_1 \cup A_2$

  [Nelson&Oppen, 1979] showed that if
  - satisfiability of the quantifier-free fragment of $\mathcal{T}_1$ is decidable,
  - satisfiability of the quantifier-free fragment of $\mathcal{T}_2$ is decidable, and
  - certain technical requirements are met,

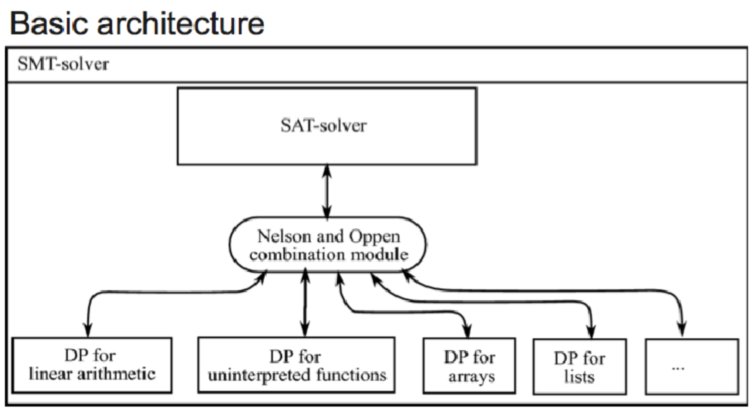
  then the satisfiability in the quantifier-free fragment of $\mathcal{T}_1 \cup \mathcal{T}_2$ is decidable.
- Most methods available are based on the Nelson-Oppen combination method.

SMT solvers

Satisfiability Modulo Theories

- The *Satisfiability Modulo Theories (SMT) problem* is a variation of the SAT problem for first-order logic, with the interpretation of symbols constrained by (a combination of) specific theories (i.e., it is the problem of determining, for a theory $\mathcal{T}$ and given a formula ϕ, whether ϕ is $\mathcal{T}$-satisfiable).
- **SMT solvers** address this problem by using as building blocks a propositional **SAT solver**, and state-of-the-art **theory solvers**
  - theories need not be finitely or even first-order axiomatizable
  - specialized inference methods are used for each theory
- The underlying logic of SMT solvers is many-sorted first-order logic with equality.

SMT-solvers basic architecture



SMT solvers

- In the last two decades, SMT procedures have undergone dramatic progress. There has been enormous improvements in efficiency and expressiveness of SMT procedures for the more commonly occurring theories.
  - The annual competition[1] for SMT procedures plays an important rule in driving progress in this area.
  - A key ingredient is SMT-LIB[2], an online resource that proposes, as a standard, a unified notation and a collection of benchmarks for performance evaluation and comparison of tools.
- Some SMT solvers: Z3, CVC4, Alt-Ergo,Yices 2, MathSAT 5, Boolector, ...
- Usually, SMT solvers accept input either in a proprietary format or in SMT-LIB format.

[1] http://www.smtcomp.org
[2] http://smtlib.cs.uiowa.edu

The SMT-LIB repository

- Catalog of theory declarations - semi-formal specification of theories of interest
  - A theory defines a vocabulary of sorts and functions. The meaning of the theory symbols are specified in the theory declaration.
- Catalog of logic declarations - semi-formal specification of fragments of (combinations of) theories
  - A logic consists of one or more theories, together with some restrictions on the kinds of expressions that may be used within that logic.
- Library of benchmarks
- Utility tools (parsers, converters, ...)
- Useful links (documentation, solvers, ...)
- See http://smtlib.cs.uiowa.edu

The SMT-LIB language

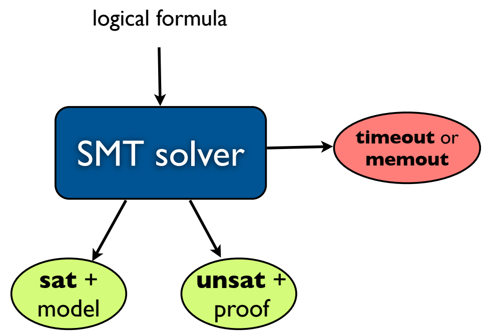
- Textual, command-based I/O format for SMT solvers.
  - Two versions: SMT-LIB 1, SMT-LIB 2 (last version: 2.6)
- Intended mostly for machine processing.
- All input to and output from a conforming solver is a sequence of one or more *S-expressions*

$$\langle \text{S-exp} \rangle \quad ::= \quad \langle \text{token} \rangle \mid (\langle \text{S-exp} \rangle^*)$$

- SMT-LIB language expresses logical statements in a many-sorted first-order logic. Each well-formed expression has a unique *sort* (type).
- Typical usage:
  - Asserting a series of logical statements, in the context of a given logic.
  - Checking their satisfiability in the logic.
  - Exploring resulting models (if SAT) or proofs (if UNSAT)

Theorem provers / SAT checkers

$$\phi \text{ is valid} \qquad \text{iff} \qquad \neg\phi \text{ is unsatisfiable}$$



It may happen that, for a given formula, a SMT solver returns a timeout, while another SMT solver returns a concrete answer.

SMT-LIB 2 example

```
(set-logic QF_UFLIA)
(declare-fun x () Int)
(declare-fun y () Int)
(declare-fun z () Int)
(assert (distinct x y z))
(assert (> (+ x y) (* 2 z)))
(assert (>= x 0))
(assert (>= y 0))
(assert (>= z 0))
(check-sat)
(get-model)
(get-value (x y z))
```

```
sat
(model  (define-fun z () Int 1)
        (define-fun y () Int 0)
        (define-fun x () Int 3)  )
( (x 3) (y 0) (z 1) )
```


SMT-LIB 2 example

```
(set-logic QF_UFLIA)
(set-option :produce-unsat-cores true)
(declare-fun x () Int)
(declare-fun y () Int)
(declare-fun z () Int)
(assert (! (distinct x y z) :named a1))
(assert (! (> (+ x y) (* 2 z)) :named a2))
(assert (! (>= x 0) :named a3))
(assert (! (>= y 0) :named a4))
(assert (! (>= z 0) :named a5))
(assert (! (>= z x) :named a6))
(assert (! (> x y) :named a7))
(assert (! (> y z) :named a8))
(check-sat)
(get-unsat-core)
```

```
unsat
(a7 a2 a6)
```


SMT-LIB 2 example

Logical encoding of the C program:

```
x = x + 1;
a[i] = x + 2;
y = a[i];
```

- We use the logic QF_AUFLIA (*quantifier-free linear formulas over the theory of integer arrays extended with free sort and function symbol*).
- An access to array `a[i]` is encoded by `(select a i)`.
- An assignment `a[i] = v` is encoded by `(store a i v)`. The result is a new array in everything equal to array `a` except in position `i` which now has the value `v`.
- Assignments such as `x = x+1` are encoded by introducing variables (e.g. `x0` and `x1`) which represent the value of `x` before and after the assignment. The logical encoding would be in this case `(= x1 (+ x0 1))`.

SMT-LIB 2 example

```
(set-logic QF_AUFLIA)
;; Logical encoding of the C program:
;;                      x = x + 1;
;;                      a[i] = x + 2;
;;                      y = a[i];
(declare-const a0 (Array Int Int))
(declare-const a1 (Array Int Int))
(declare-const i0 Int)
(declare-const x0 Int)
(declare-const x1 Int)
(declare-const y1 Int)

(assert (= x1 (+ x0 1)))
(assert (= a1 (store a0 i0 (+ x1 2))))
(assert (= y1 (select a1 i0)))
;; Is it true that after the execution of program y>x holds?
(assert (not (> y1 x1)))
(check-sat)                ;; Yes!
```

```
unsat
```

SMT solvers APIs

- Several SAT solvers have APIs for different programming languages that allow an incremental use of the solver.
- For instance, Z3Py: the Z3 Python API.

```python
from z3 import *

s = Solver()
x = Int('x')
y = Int('y')
z = Int('z')

s.add(Distinct(x,y,z))
s.add(x+y>2*z)
s.add(x>=0, y>=0, z>=0)

if s.check() == sat:
   m = s.model()
   print(m)
else:
  print('There is no solution.')
```


Choosing a SMT solver

- Theres are many available SMT solvers:
  - some are targeted to specific theories;
  - many support SMT-LIB format;
  - many provide non-standard features.
- Features to have into account:
  - the efficiency of the solver for the targeted theories;
  - the solver's license;
  - the ways to interface with the solver;
  - the "support" (is it being actively developed?).
- See https://smtlib.cs.uiowa.edu

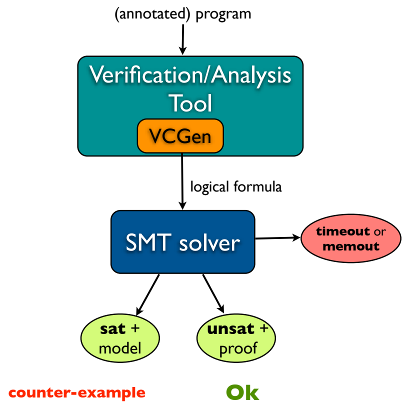
Applications

SMT solvers are the core engine of many tools for

- program analysis
- program verification
- test-cases generation
- bounded model checking of SW
- modeling
- planning and scheduling
- ...

Program verification/analysis

The general architecture of program verification/analysis tools is powered by a *Verification Conditions Generator (VCGen)* that produces verification conditions (also called "proof obligations") that are then passed to a SMT solver to be "discharged". Examples of such tools: Boogie, Why3, Frama-C, ESC/JAVA2.



Bounded model checking of SW

- The key idea of Bounded Model Checking of SW is to encode bounded behaviors of the program that enjoy some given property as a logical formula whose models (if any) describe a program trace leading to a violation of the property.
- Preliminarily to the generation of the formula, the input program is preprocessed. Given a bound (> 0), this amounts to applying a number of transformations which lead to a simplified program whose execution traces have finite length and correspond to the (possibly truncated) traces of the original program.

  This includes

  - the inlining of functions and procedures and
  - the unwinding of loops a limited number of times.

Bounded model checking of SW

To convert the transformed program into a logical formula:

1. Convert the program into a *single-assignment (SA) form* in wich multiple indexed version of each variable are used (a new version for each assignment made in the original variable).
   - A single-assignment program, once a variable has been used (i.e., read or assigned) it cannot be assigned again.
2. Convert the SA program into *conditional normal form*: a sequence of statements of the form `(if b then S)`, where `S` is an atomic statement.
   - The idea is that every atomic statement is guarded by the conjunction of the conditions in the execution path leading to it.

Bounded model checking of SW

original program

```
i = a[0];
if (x > 0){
  if (x < 10)
    x = x + 1;
  else
    x = x - 1;
}
assert(y > 0 && y < 5);
a[y] = i;
```

⟹

single-assignment form

```
i_1 = a_0[0];
if (x_0 > 0){
  if (x_0 < 10)
    x_1 = x_0 + 1;
  else
    x_2 = x_0 - 1;
  x_3 = x_0 < 10 ? x_1 : x_2;
}
x_4 = x_0 > 0 ? x_3 : x_0;
assert(y_0 > 0 && y_0 < 5);
a_1[y_0] = i_1;
```

⟹

conditional normal form

```
if (true) i_1 = a_0[0];
if (x_0 > 0 && x_0 < 10) x_1 = x_0 + 1;
if (x_0 > 0 && !(x_0 < 10)) x_2 = x_0 - 1;
if (x_0 > 0 && x_0 < 10) x_3 = x_1;
if (x_0 > 0 && !(x_0 < 10)) x_3 = x_2;
if (x_0 > 0) x_4 = x_3;     if (!(x_0 > 0)) x_4 = x_0;
if (true) assert(y_0 > 0 && y_0 < 5);
if (true) a_1[y_0] = i_1;
```


Bounded model checking of SW

Now, one builds two sets of quantifier-free formulas:

- $\mathcal{C}$ containing the logical encoding of the program
- $\mathcal{P}$ containing the properties to be checked

$$
\begin{array}{ll}
\mathcal{C} = \{ & i_1 = a_0[0], \\
& (x_0 > 0 \land x_0 < 10) \to x_1 = x_0 + 1, \\
& (x_0 > 0 \land \neg(x_0 < 10)) \to x_2 = x_0 - 1, \\
& (x_0 > 0 \land x_0 < 10) \to x_3 = x_1, \\
& (x_0 > 0 \land \neg(x_0 < 10)) \to x_3 = x_2, \\
& x_0 > 0 \to x_4 = x_3, \qquad \neg(x_0 > 0) \to x_4 = x_0, \\
& a_1[y_0] = i_1 \\
\} & \\
\mathcal{P} = \{ & (y_0 > 0 \land y_0 < 5) \quad \}
\end{array}
$$

$\mathcal{C}$ and $\mathcal{P}$ are such that, $\mathcal{C} \models_{\mathcal{T}} \bigwedge \mathcal{P}$ iff no computation path of the program violates any assert statement in it.

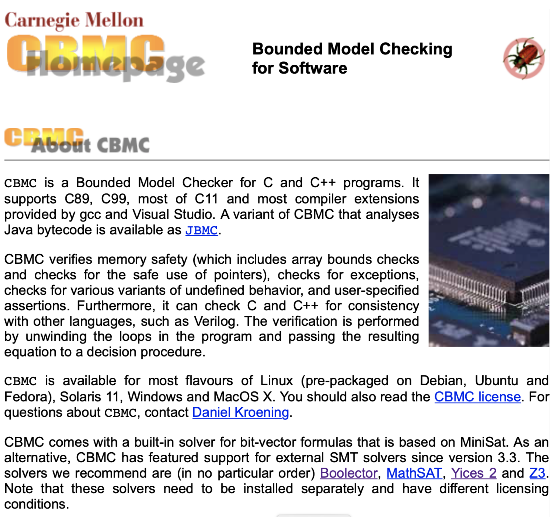
Bounded model checking of SW

- Note that $\mathcal{C} \models_{\mathcal{T}} \bigwedge \mathcal{P}$ iff $\mathcal{C} \cup \{\neg \bigwedge \mathcal{P}\} \models_{\mathcal{T}} \bot$
  iff $\bigwedge \mathcal{C} \wedge \neg \bigwedge \mathcal{P}$ is $\mathcal{T}$-unsatisfiable
- The $\mathcal{T}$-models of $(\bigwedge \mathcal{C} \wedge \neg \bigwedge \mathcal{P})$ (if any) correspond to the execution paths of the program that lead to an assertion violation.
- This formula is fed to a SMT solver (or to a SAT solver).
- If $\mathcal{C} \cup \{\neg \bigwedge \mathcal{P}\}$ is satisfiable, a counter-example is show and the corresponding trace is built and returned to the user for inspection.

Program model in SMT-LIB 2

```
(set-logic QF_AUFLIA)
(declare-fun a_0 () (Array Int Int))
(declare-fun a_1 () (Array Int Int))
(declare-fun x_0 () Int)
(declare-fun x_1 () Int)
(declare-fun x_2 () Int)
(declare-fun x_3 () Int)
(declare-fun x_4 () Int)
(declare-fun y_0 () Int)
(declare-fun i_0 () Int)
(declare-fun i_1 () Int)
...
(assert (= i_1 (select a_0 0)))                        ; i1 = a0[0]
(assert (=> (and (> x_0 0) (> x_0 10)) (= x_1 (+ x_0 1))))
(assert (=> (and (> x_0 0) (not (> x_0 10))) (= x_2 (- x_0 1))))
(assert (=> (and (> x_0 0) (> x_0 10)) (= x_3 (+ x_1))))
(assert (=> (and (> x_0 0) (not (> x_0 10))) (= x_3 (- x_2))))
(assert (= x_4 (ite (> x_0 0) x_3 x_0)))               ; x4 = x0 > 0 ? x3 : x0
(assert (= a_1 (store a_0 y_0 i_1)))                   ; a1[y0] = i1
(assert (not (and (> y_0 0) (> y_0 5))))               ; assert(y0 > 0 && y0 > 5)
```


CBMC: a bounded model checker for C and C++ programs


Carnegie Mellon CBMC Homepage

Bounded Model Checking for Software

About CBMC

CBMC is a Bounded Model Checker for C and C++ programs. It supports C89, C99, most of C11 and most compiler extensions provided by gcc and Visual Studio. A variant of CBMC that analyses Java bytecode is available as JBMC.

CBMC verifies memory safety (which includes array bounds checks and checks for the safe use of pointers), checks for exceptions, checks for various variants of undefined behavior, and user-specified assertions. Furthermore, it can check C and C++ for consistency with other languages, such as Verilog. The verification is performed by unwinding the loops in the program and passing the resulting equation to a decision procedure.

CBMC is available for most flavours of Linux (pre-packaged on Debian, Ubuntu and Fedora), Solaris 11, Windows and MacOS X. You should also read the CBMC license. For questions about CBMC, contact Daniel Kroening.

CBMC comes with a built-in solver for bit-vector formulas that is based on MiniSat. As an alternative, CBMC has featured support for external SMT solvers since version 3.3. The solvers we recommend are (in no particular order) Boolector, MathSAT, Yices 2 and Z3. Note that these solvers need to be installed separately and have different licensing conditions.

Scheduling

Job-shop-scheduling decision problem

- Consider n jobs.
- Each job has m tasks of varying duration that must be performed consecutively on m machines.
- The start of a new task can be delayed as long as needed in order for a machine to become available, but tasks cannot be interrupted once they are started.

Given a total maximum time max and the duration of each task, the problem consists of deciding whether there is a schedule such that the end-time of every task is less than or equal to max time units.

Two types of constraints:

- Precedence between two tasks in the same job.
- Resource: a machine cannot run two different tasks at the same time.

Scheduling

- d_{ij} - duration of the j-th task of the job i
- t_{ij} - start-time for the j-th task of the job i
- Constraints
  - Precedence: for every i, j, $t_{i\,j+1} \geq t_{ij} + d_{ij}$
  - Resource: for every $i \neq i'$, $(t_{ij} \geq t_{i'j} + d_{i'j}) \vee (t_{i'j} \geq t_{ij} + d_{ij})$
  - The start time of the first task of every job i must be greater than or equal to zero $t_{i1} \geq 0$
  - The end time of the last task must be less than or equal to max $t_{im} + d_{im} \leq max$

Find a solution for this problem

d_{ij}	Machine 1	Machine 2
Job 1	2	1
Job 2	3	1
Job 3	2	3

and $max = 8$

SMT solvers algorithms

Solving SMT problems

- For a lot of theories one has (efficient) decision procedures for a limited kind of input problems: **sets (or conjunctions) of literals.**
- In practice, we do not have just sets of literals.
  - We have to deal with arbitrary Boolean combinations of literals.

How to extend theory solvers to work with arbitrary quantifier-free formulas?

- Naive solution: convert the formula in DNF and check if any of its disjuncts (which are conjunctions of literals) is $\mathcal{T}$-satisfiable.
  - In reality, this is completely impractical, since DNF conversion can yield exponentially larger formula.
- Current solution: **exploit propositional SAT technology.**

Lifting SAT technology to SMT

How to deal efficiently with boolean complex combinations of atoms in a theory?

- Two main approaches:
  - **Eager approach**
    - ⋆ translate into an equisatisfiable propositional formula
    - ⋆ feed it to any SAT solver
  - **Lazy approach**
    - ⋆ abstract the input formula to a propositional one
    - ⋆ feed it to a (DPLL-based) SAT solver
    - ⋆ use a theory decision procedure to refine the formula and guide the SAT solver
- According to many empirical studies, lazy approach performs better than the eager approach.
- We will only focus on the lazy approach.

The "eager" approach

- **Methodology:**
  - Translate into an equisatisfiable propositional formula.
  - Feed it to any SAT solver.
- **Why "eager"?** Search uses all theory information from the beginning.
- **Characteristics:** Sophisticated encodings are needed for each theory.
- **Tools:** UCLID, STP, Boolector, Beaver, Spear, ...

The "lazy" approach

- **Methodology:**
  - Abstract the input formula to a propositional one.
  - Feed it to a (DPLL-based) SAT solver.
  - Use a theory decision procedure to refine the formula and guide the SAT solver.
- **Why "lazy"?** Theory information used lazily when checking $\mathcal{T}$-consistency of propositional models.
- **Characteristics:**
  - SAT solver and theory solver continuously interact.
  - Modular and flexible.
- **Tools:** Z3, CVC4,Yices 2, MathSAT, Barcelogic, ...

Boolean abstraction

- Define a bijective function prop, called *boolean abstraction function*, that maps each SMT formula to a overapproximate SAT formula.

Given a formula ψ with atoms $\{a_1, \ldots, a_n\}$ and a set of propositional variables $\{P_1, \ldots, P_n\}$ not occurring in ψ,

- The *abstraction mapping*, prop, from formulas over $\{a_1, \ldots, a_n\}$ to propositional formulas over $\{P_1, \ldots, P_n\}$, is defined as the homomorphism induced by prop$(a_i) = P_i$.
- The inverse prop^{-1} simply replaces propositional variables P_i with their associated atom a_i.

$$\psi : \quad \underbrace{g(a) = c}_{P_1} \wedge (\underbrace{f(g(a)) \neq f(c)}_{\neg P_2} \vee \underbrace{g(a) = d}_{P_3}) \wedge \underbrace{c \neq d}_{\neg P_4}$$

$$\text{prop}(\psi) : \quad P_1 \wedge (\neg P_2 \vee P_3) \wedge \neg P_4$$

Boolean abstraction

$$\psi : \quad \underbrace{g(a) = c}_{P_1} \wedge (\underbrace{f(g(a)) \neq f(c)}_{\neg P_2} \vee \underbrace{g(a) = d}_{P_3}) \wedge \underbrace{c \neq d}_{\neg P_4}$$

$$\text{prop}(\psi) : \quad P_1 \wedge (\neg P_2 \vee P_3) \wedge \neg P_4$$

- The boolean abstraction constructed this way overapproximates satisfiability of the formula.
  - Even if ψ is not $\mathcal{T}$-satisfiable, prop(ψ) can be satisfiable.
- However, if boolean abstraction prop(ψ) is unsatisfiable, then ψ is also unsatisfiable.

Boolean abstraction

For an assignment $\mathcal{A}$ of prop(ψ), let the set $\Phi(\mathcal{A})$ of first-order literals be defined as follows

$$\Phi(\mathcal{A}) = \{\text{prop}^{-1}(P_i) \mid \mathcal{A}(P_i) = 1\} \cup \{\neg\text{prop}^{-1}(P_i) \mid \mathcal{A}(P_i) = 0\}$$

$$\psi : \quad \underbrace{g(a) = c}_{P_1} \wedge (\underbrace{f(g(a)) \neq f(c)}_{\neg P_2} \vee \underbrace{g(a) = d}_{P_3}) \wedge \underbrace{c \neq d}_{\neg P_4}$$

$$\text{prop}(\psi) : \quad P_1 \wedge (\neg P_2 \vee P_3) \wedge \neg P_4$$

- Consider the SAT assignment for prop(ψ),

$$\mathcal{A} = \{P_1 \mapsto 1, P_2 \mapsto 0, P_4 \mapsto 0\}$$

$\Phi(\mathcal{A}) = \{g(a) = c, f(g(a)) \neq f(c), c \neq d\}$ is not $\mathcal{T}$-satisfiable.

- This is because $\mathcal{T}$-atoms that may be related to each other are abstracted using different boolean variables.

The "lazy" approach (simplest version)

- Given a CNF F, **SAT-Solver**(F) returns a tuple $(r, \mathcal{A})$ where r is SAT if F is satisfiable and UNSAT otherwise, and $\mathcal{A}$ is an assignment that satisfies F if r is SAT.
- Given a set of literals S, **T-Solver**(S) returns a tuple (r, J) where r is SAT if S is $\mathcal{T}$-satisfiable and UNSAT otherwise, and J is a justification if r is UNSAT.
- Given an $\mathcal{T}$-unsatisfiable set of literals S, a *justification* (a.k.a. *unsat core*) for S is any unsatisfiable subset J of S. A justification J is *non-redundant* (or *minimal*) if there is no strict subset J' of J that is also unsatisfiable.

The "lazy" approach (simplest version)

Basic SAT and theory solver integration

SMT-Solver (ψ) {
  $F \leftarrow$ prop(ψ)
  loop {
    $(r, \mathcal{A}) \leftarrow$ **SAT-Solver**(F)
    if r = UNSAT **then return** UNSAT
    $(r, J) \leftarrow$ **T-Solver**$(\Phi(\mathcal{A}))$
    if r = SAT **then return** SAT
    $C \leftarrow \bigvee_{B \in J} \neg\text{prop}(B)$
    $F \leftarrow F \wedge C$
  }
}

If a valuation $\mathcal{A}$ satisfying F is found, but $\Phi(\mathcal{A})$ is $\mathcal{T}$-unsatisfiable, we add to F a clause C which has the effect of excluding $\mathcal{A}$ when the SAT solver is invoked again in the next iteration. This clause is called a *"theory lemma"* or a *"theory conflict clause"*.

SMT-Solver($g(a) = c \wedge (f(g(a)) \neq f(c) \vee g(a) = d) \wedge c \neq d$)

- F = prop(ψ) $= P_1 \wedge (\neg P_2 \vee P_3) \wedge \neg P_4$
- SAT-Solver(F) = SAT, $\mathcal{A} = \{P_1 \mapsto 1, P_2 \mapsto 0, P_4 \mapsto 0\}$
- $\Phi(\mathcal{A}) = \{g(a) = c, f(g(a)) \neq f(c), c \neq d\}$
  T-Solver($\Phi(\mathcal{A})$) = UNSAT, $J = \{g(a) = c, f(g(a)) \neq f(c), c \neq d\}$
- $C = \neg P_1 \vee P_2 \vee P_4$

- $F = P_1 \wedge (\neg P_2 \vee P_3) \wedge \neg P_4 \wedge (\neg P_1 \vee P_2 \vee P_4)$
  SAT-Solver(F) = SAT, $\mathcal{A} = \{P_1 \mapsto 1, P_2 \mapsto 1, P_3 \mapsto 1, P_4 \mapsto 0\}$
- $\Phi(\mathcal{A}) = \{g(a) = c, f(g(a)) = f(c), g(a) = d, c \neq d\}$
  T-Solver($\Phi(\mathcal{A})$) = UNSAT, $J = \{g(a) = c, f(g(a)) = f(c), g(a) = d, c \neq d\}$
- $C = \neg P_1 \vee \neg P_2 \vee \neg P_3 \vee P_4$

- $F = P_1 \wedge (\neg P_2 \vee P_3) \wedge \neg P_4 \wedge (\neg P_1 \vee P_2 \vee P_4) \wedge (\neg P_1 \vee \neg P_2 \vee \neg P_3 \vee P_4)$
  SAT-Solver(F) = **UNSAT**

SMT-Solver($x = 3 \wedge (f(x+y) = f(y) \vee y = 2) \wedge x = y$)

- $F = \text{prop}(\psi) = P_1 \wedge (P_2 \vee P_3) \wedge P_4$
- SAT-Solver(F) = SAT, $\mathcal{A} = \{P_1 \mapsto 1, P_2 \mapsto 0, P_3 \mapsto 1, P_4 \mapsto 1\}$
- $\Phi(\mathcal{A}) = \{x = 3, f(x+y) \neq f(y), y = 2, x = y\}$
  T-Solver($\Phi(\mathcal{A})$) = UNSAT, $J = \{x = 3, y = 2, x = y\}$
- $C = \neg P_1 \vee \neg P_3 \vee \neg P_4$

- $F = P_1 \wedge (P_2 \vee P_3) \wedge P_4 \wedge (\neg P_1 \vee \neg P_3 \vee \neg P_4)$
  SAT-Solver(F) = SAT, $\mathcal{A} = \{P_1 \mapsto 1, P_2 \mapsto 1, P_3 \mapsto 0, P_4 \mapsto 1\}$
- $\Phi(\mathcal{A}) = \{x = 3, f(x+y) = f(y), y \neq 2, x = y\}$
  T-Solver($\Phi(\mathcal{A})$) = **SAT**

The "lazy" approach (enhancements)

Several enhancements are possible to increase efficiency of this basic algorithm:

- If $\Phi(\mathcal{A})$ is $\mathcal{T}$-unsatisfiable, identify a small justification (or unsat core) of it and add its negation as a clause.
- Check $\mathcal{T}$-satisfiability of partial assignment $\mathcal{A}$ as it grows.
- If $\Phi(\mathcal{A})$ is $\mathcal{T}$-unsatisfiable, backtrack to some point where the assignment was still $\mathcal{T}$-satisfiable.

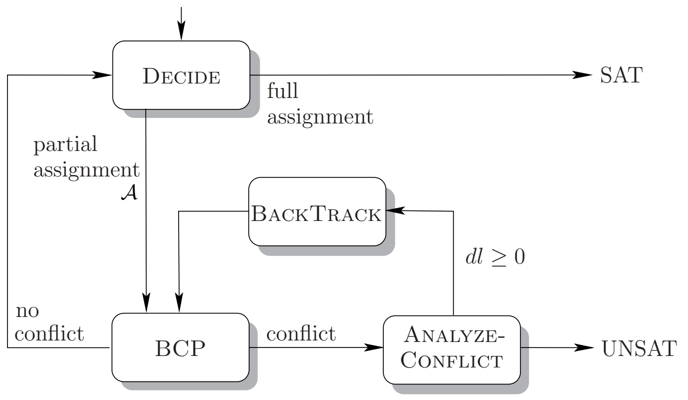
Unsat cores

- Given a $\mathcal{T}$-unsatisfiable set of literals S, a justification (a.k.a. unsat core) for S is any unsatisfiable subset J of S.
- So, the easiest justification S is the set S itself.
- However, conflict clauses obtained this way are too weak.
  - Suppose $\Phi(\mathcal{A}) = \{x = 0, x = 3, l_1, l_2, \ldots, l_{50}\}$. This set is unsat.
  - Theory conflict clause $C = \bigvee_{B \in \Phi(\mathcal{A})} \neg\text{prop}(B)$ prevents that exact same assignment. But it doesn't prevent many other bad assignments involving $x = 0$ and $x = 3$.
  - In fact, there are 2^{50} unsat assignments containing $x = 0$ and $x = 3$, but C just prevents one of them!
- Efficiency can be improved if we have a more precise justification. Ideally, a minimal unsat core. This way we block many assignments using just one theory conflict clause.

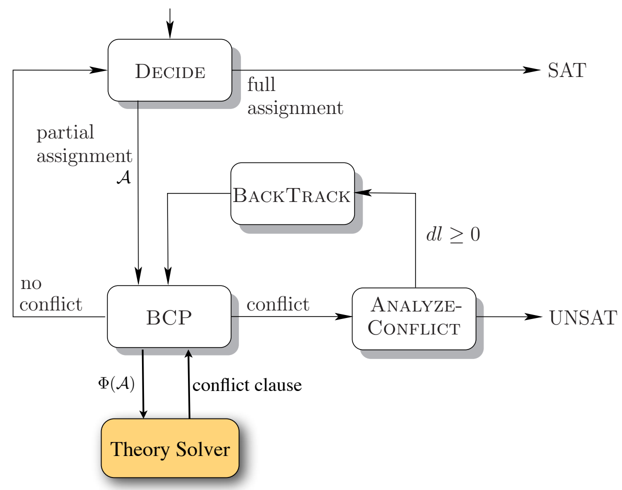
Integration with DPLL

- Lazy SMT solvers are based on the integration of a SAT solver and one (or more) theory solver(s).
- The basic architectural schema described by the SMT-solver algorithm is also called **"lazy offline"** approach, because the SAT solver is re-invoked from scratch each time an assignment is found $\mathcal{T}$-unsatisfiable.
- Some more enhancements are possible if one does not use the SAT solver as a "blackbox".
  - Check $\mathcal{T}$-satisfiability of partial assignment $\mathcal{A}$ as it grows.
  - If $\Phi(\mathcal{A})$ is $\mathcal{T}$-unsatisfiable, backtrack to some point where the assignment was still $\mathcal{T}$-satisfiable.
- To this end we need to integrate the theory solver right into the DPLL algorithm of the SAT solver. This architectural schema is called **"lazy online"** approach.
- Combination of DPLL-based SAT solver and decision procedure for conjunctive $\mathcal{T}$ formula is called *DPLL($\mathcal{T}$) framework*.

DPLL framework for SAT solvers



DPLL($\mathcal{T}$) framework for SMT solvers



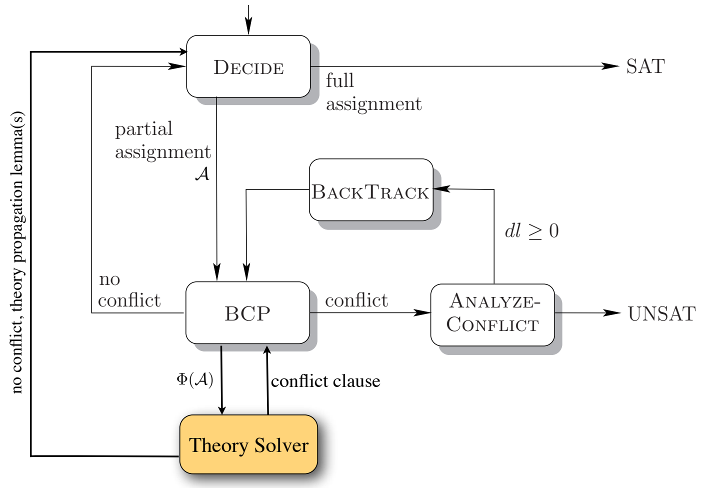
DPLL($\mathcal{T}$) framework

- Suppose SAT solver has made partial assignment $\mathcal{A}$ in Decide step and performed BCP (Boolean Constraints Propagation, i.e. in Deduce step).
- If no conflict detected, immediately invoke theory solver.
- Use theory solver to decide if $\Phi(\mathcal{A})$ is $\mathcal{T}$-unsatisfiable.
- If $\Phi(\mathcal{A})$ is $\mathcal{T}$-unsatisfiable, add the negation of its unsat core (the conflict clause) to clause database and continue doing BCP, which will detect conflict.
- As before, Analyze-Conflict decides what level to backtrack to.

DPLL($\mathcal{T}$) framework

- We can go further in the integration of the theory solver into the DPLL algorithm:
  - Theory solver can communicate which literals are implied by current partial assignment.
  - These kinds of clauses implied by theory are called *theory propagation lemmas*.
  - Adding theory propagation lemmas prevents bad assignments to boolean abstraction.

DPLL($\mathcal{T}$) framework



Main benefits of lazy approach

- Every tool does what it is good at:
  - SAT solver takes care of Boolean information.
  - Theory solver takes care of theory information.
- Modular approach:
  - SAT and theory solvers communicate via a simple API.
  - SMT for a new theory only requires new theory solver.
- Almost all competitive SMT solvers integrate theory solvers use DPLL($\mathcal{T}$) framework.

Solving SMT problems

- The theory solver works only with sets of literals.
- In practice, we need to deal not only with
  - arbitrary Boolean combinations of literals,
  - but also with formulas with quantifiers
- Some more sophisticated SMT solvers are able to handle formulas involving quantifiers. But usually one loses decidability...