Cálculo
de
Programas

Class T01 — 19-Sep

Introduction to the course.

Website: **`https://haslab.github.io/CP`**

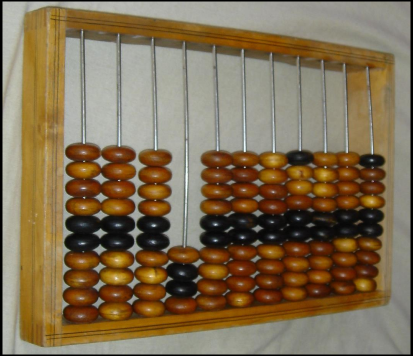
Other administrative details.

J.N. Oliveira

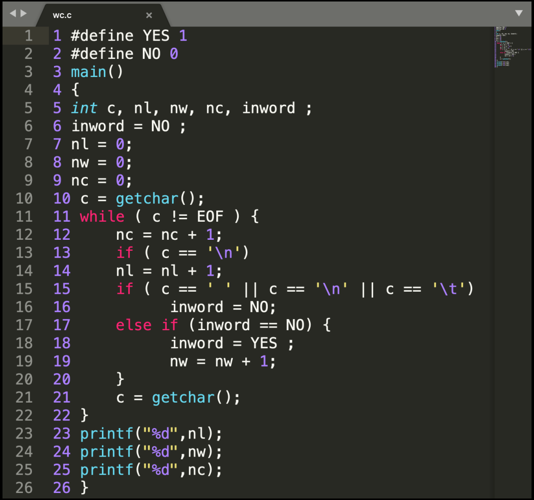
Programming by Calculation

Program versus Calculus

- **Calculus** — abacus **pebble**
- **Program** — *pro* (‘before’) + *graphein* (‘write’)

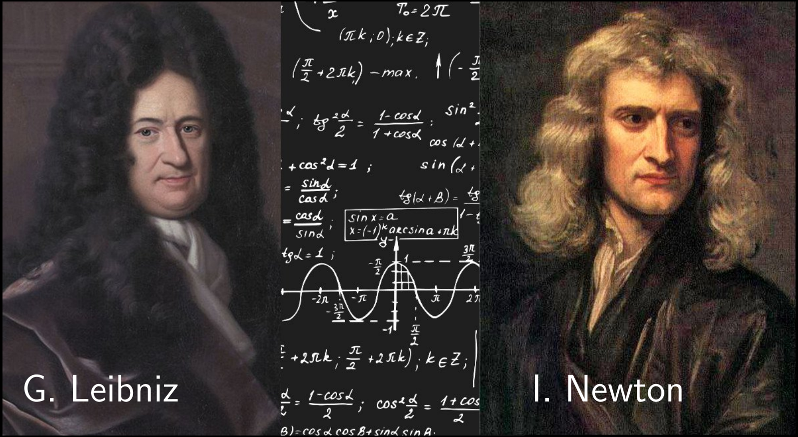


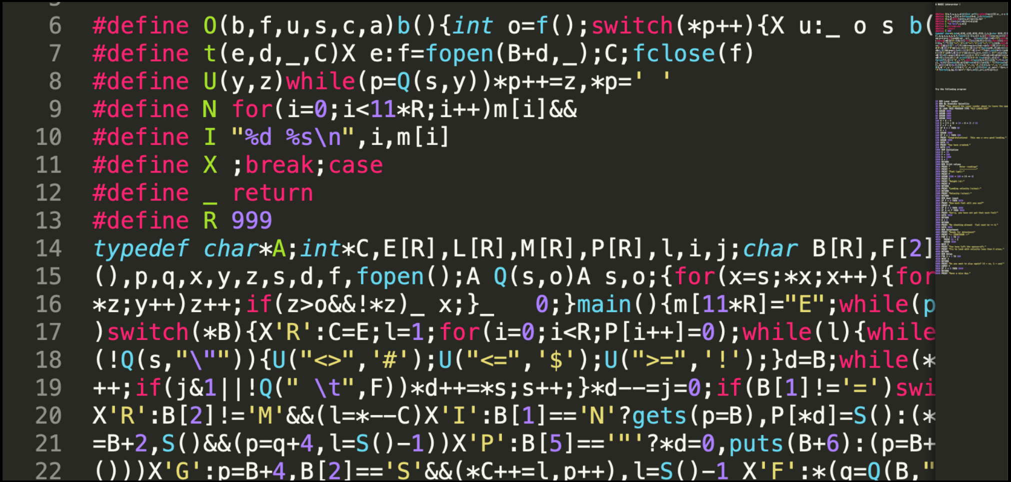
Programs...



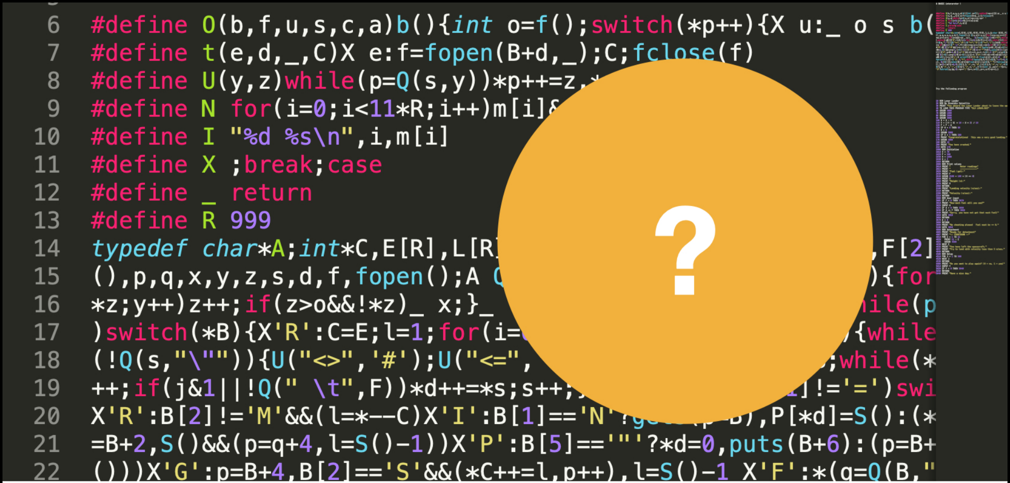
```
1 #define YES 1
2 #define NO 0
3 main()
4 {
5 int c, nl, nw, nc, inword ;
6 inword = NO ;
7 nl = 0;
8 nw = 0;
9 nc = 0;
10 c = getchar();
11 while ( c != EOF ) {
12     nc = nc + 1;
13     if ( c == '\n')
14     nl = nl + 1;
15     if ( c == ' ' || c == '\n' || c == '\t')
16           inword = NO;
17     else if (inword == NO) {
18           inword = YES ;
19           nw = nw + 1;
20     }
21     c = getchar();
22 }
23 printf("%d",nl);
24 printf("%d",nw);
25 printf("%d",nc);
26 }
```

Calculus





```
#define O(b,f,u,s,c,a)b(){int o=f();switch(*p++){X u:_ o s b(
#define t(e,d,_,C)X e:f=fopen(B+d,_);C;fclose(f)
#define U(y,z)while(p=Q(s,y))*p++=z,*p=' '
#define N for(i=0;i<11*R;i++)m[i]&&
#define I "%d %s\n",i,m[i]
#define X ;break;case
#define _ return
#define R 999
typedef char*A;int*C,E[R],L[R],M[R],P[R],l,i,j;char B[R],F[2]
(),p,q,x,y,z,s,d,f,fopen();A Q(s,o)A s,o;{for(x=s;*x;x++){for
*z;y++)z++;if(z>o&&!*z)_ x;}_   0;}main(){m[11*R]="E";while(p
)switch(*B){X'R':C=E;l=1;for(i=0;i<R;P[i++]=0);while(l){while
(!Q(s,"\"")){U("<>",'#');U("<=",'$');U(">=",'!');}d=B;while(*
++;if(j&1||!Q(" \t",F))*d++=*s;s++;}*d--=j=0;if(B[1]!='=')swi
X'R':B[2]!='M'&&(l=*--C)X'I':B[1]=='N'?gets(p=B),P[*d]=S():(*
=B+2,S()&&(p=q+4,l=S()-1))X'P':B[5]=='"'?*d=0,puts(B+6):(p=B+
()))X'G':p=B+4,B[2]=='S'&&(*C++=l,p++),l=S()-1 X'F':*(q=Q(B,"
```

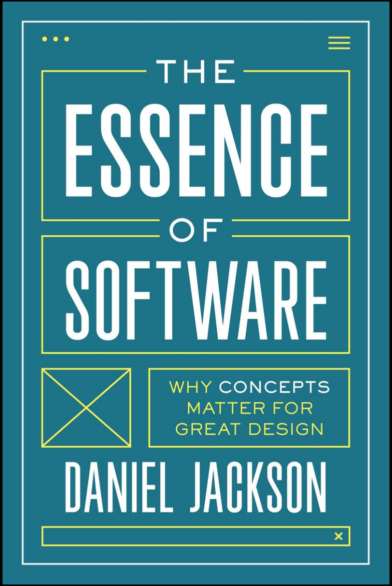


```
#define O(b,f,u,s,c,a)b(){int o=f();switch(*p++){X u:_ o s b(
#define t(e,d,_,C)X e:f=fopen(B+d,_);C;fclose(f)
#define U(y,z)while(p=Q(s,y))*p++=z,
#define N for(i=0;i<11*R;i++)m[i]&
#define I "%d %s\n",i,m[i]
#define X ;break;case
#define _ return
#define R 999
typedef char*A;int*C,E[R],L[R                              ,F[2]
(),p,q,x,y,z,s,d,f,fopen();A Q                             ){for
*z;y++)z++;if(z>o&&!*z)_ x;}_                              hile(p
)switch(*B){X'R':C=E;l=1;for(i=                            {while
(!Q(s,"\"")){U("<>",'#');U("<=",                           ;while(*
++;if(j&1||!Q(" \t",F))*d++=*s;s++;                        ]!='=')swi
X'R':B[2]!='M'&&(l=*--C)X'I':B[1]=='N'?                    ),P[*d]=S():(*
=B+2,S()&&(p=q+4,l=S()-1))X'P':B[5]=='"'?*d=0,puts(B+6):(p=B+
()))X'G':p=B+4,B[2]=='S'&&(*C++=l,p++),l=S()-1 X'F':*(q=Q(B,"
```

Software as a problem

Software

- Process — ☑
- Product — ?

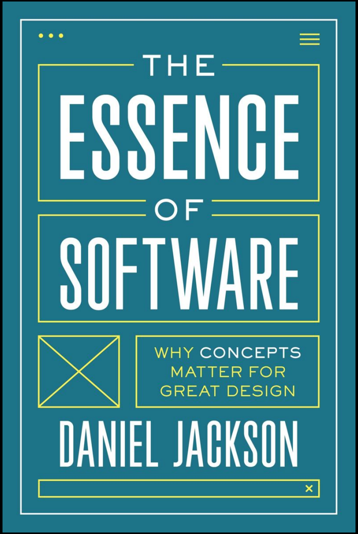


THE ESSENCE OF SOFTWARE

WHY CONCEPTS MATTER FOR GREAT DESIGN

DANIEL JACKSON

"(...) The best services revolve around **a small number of concepts** that are **well designed and easy** (...) **to understand and use**, and their innovations often involve simple but compelling new concepts."



In

The Essence of Software

by

Prof. Daniel Jackson, MIT

(2021)

… small number

… small number

… well defined

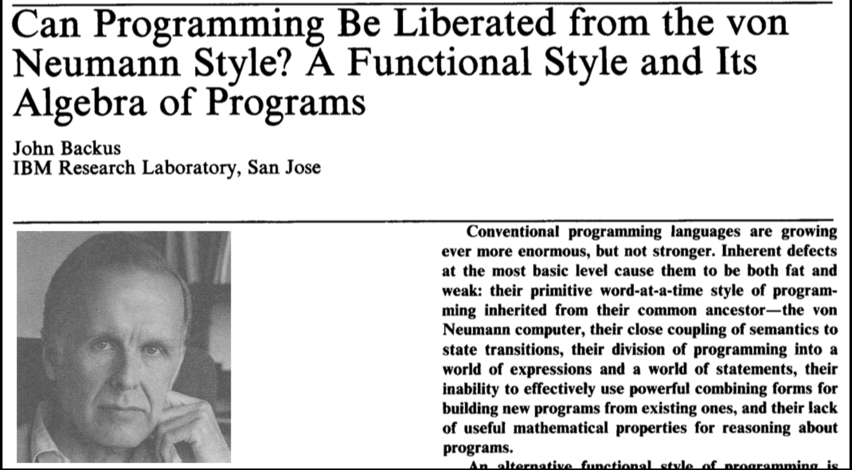
… small number

… well defined

… easy to understand

Urgently needed in software design

John Backus – Turing Award (1978)



Can Programming Be Liberated from the von Neumann Style? A Functional Style and Its Algebra of Programs

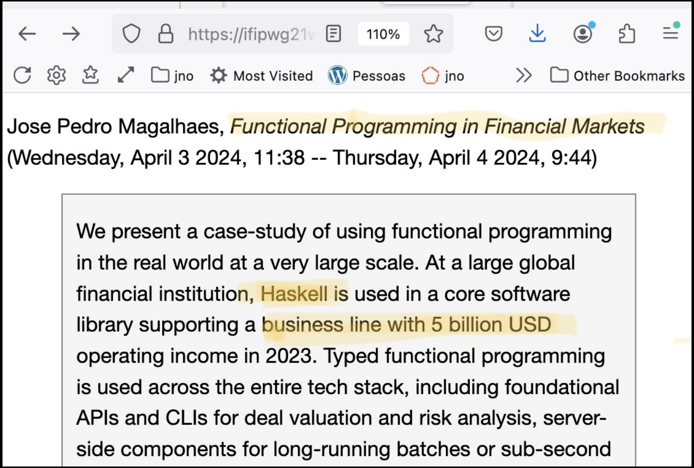
John Backus
IBM Research Laboratory, San Jose

Conventional programming languages are growing ever more enormous, but not stronger. Inherent defects at the most basic level cause them to be both fat and weak: their primitive word-at-a-time style of programming inherited from their common ancestor—the von Neumann computer, their close coupling of semantics to state transitions, their division of programming into a world of expressions and a world of statements, their inability to effectively use powerful combining forms for building new programs from existing ones, and their lack of useful mathematical properties for reasoning about programs.

An alternative functional style of programming is

Functional Programming

FP = $$$...



Jose Pedro Magalhaes, *Functional Programming in Financial Markets* (Wednesday, April 3 2024, 11:38 -- Thursday, April 4 2024, 9:44)

We present a case-study of using functional programming in the real world at a very large scale. At a large global financial institution, Haskell is used in a core software library supporting a business line with 5 billion USD operating income in 2023. Typed functional programming is used across the entire tech stack, including foundational APIs and CLIs for deal valuation and risk analysis, server-side components for long-running batches or sub-second

Part I

Motivation — back to the late 1990s

From a mobile phone manufacturer

(...) For each **list of calls** *stored in the mobile phone (e.g. numbers dialled, SMS messages, lost calls), the* **store** *operation should work in a way such that* **(a)** *the more recently a* **call** *is made the more accessible it is;* **(b)** *no number appears twice in a list;* **(c)** *only the last 10 entries in each list are stored.*

From a mobile phone manufacturer

```
store ::  Call -> [Call] -> [Call]
store c l = take 10 (nub (c:l))
```

From a mobile phone manufacturer

```
store ::  Call -> [Call] -> [Call]
```

$$\texttt{store c l = }\underbrace{\texttt{take 10}}_{(c)}\texttt{ (}\underbrace{\texttt{nub}}_{(b)}\texttt{ }\underbrace{\texttt{(c:l)}}_{(a)}\texttt{)}$$

Compare with ...

```
public void store10(string phoneNumber)
{
  System.Collections.ArrayList auxList =
      new System.Collections.ArrayList();
  auxList.Add(phoneNumber);
  auxList.AddRange(
      this.filteratmost9(phoneNumber) );
  this.callList = auxList;
}
```

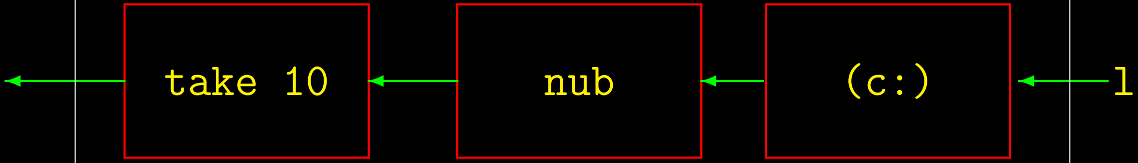
+ `filteratmost9` (next slide)

Compare with ...

```
public System.Collections.ArrayList filteratmost9(string n)
{
  System.Collections.ArrayList retList =
      new System.Collections.ArrayList();
      int i=0, m=0;
  while((i < this.callList.Count) && (m < 9))
  {
      if ((string)this.callList[i] != n)
      {
          retList.Add(this.callList[i]);
          m++;
      }
      i++;
  }
  return retList;
}
```

From a mobile phone manufacturer

`store c :: [Call] -> [Call]`



take 10

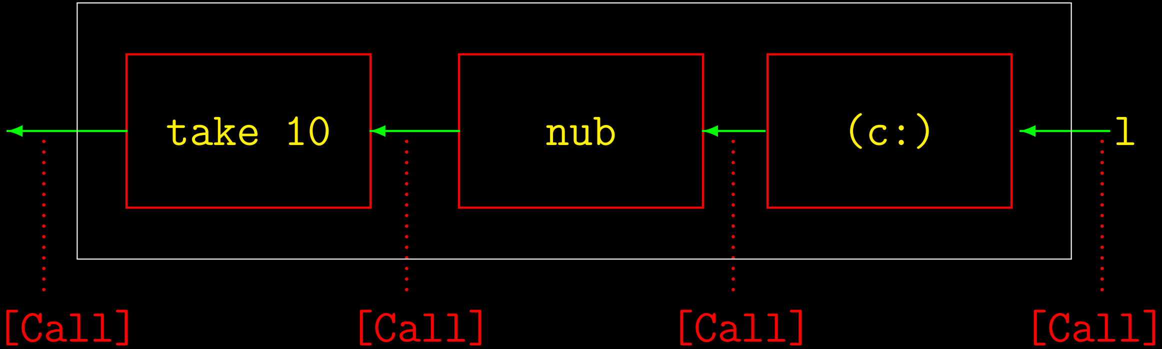
nub

(c:)

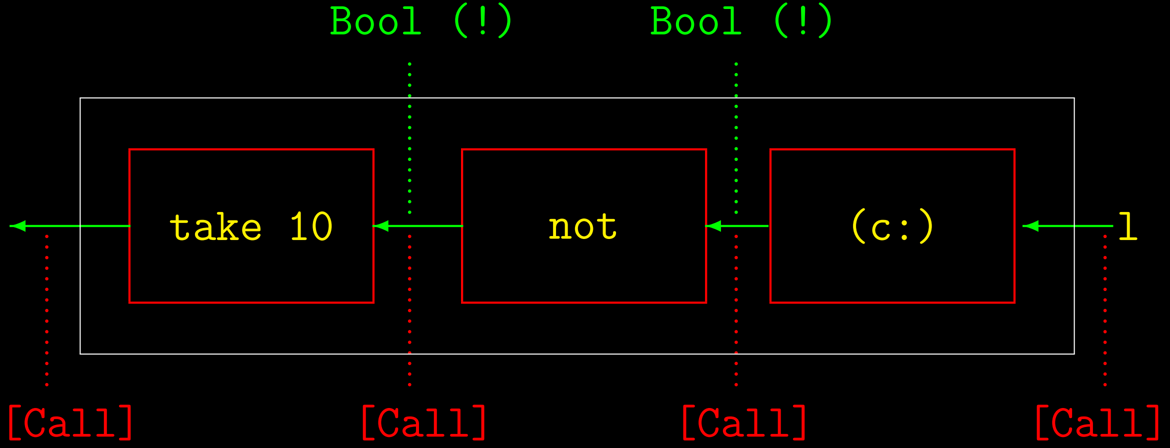
l

From a mobile phone manufacturer

`store c :: [Call] -> [Call]`



Uups!



Bool (!)

Bool (!)

take 10

not

(c:)

l

[Call]

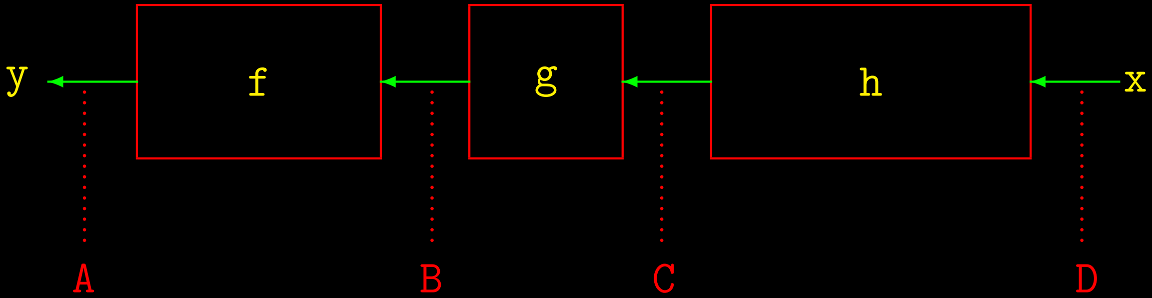
[Call]

[Call]

[Call]

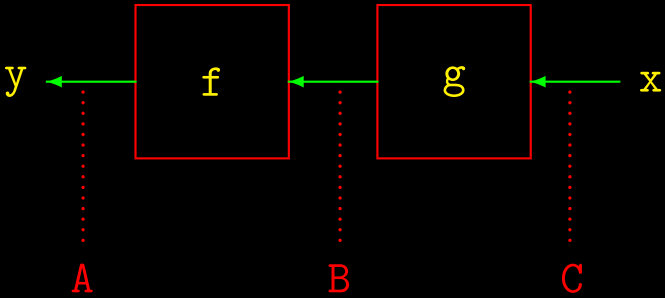
In general

`y = f(g(h x))`



In general

```
y = f(g x)
```

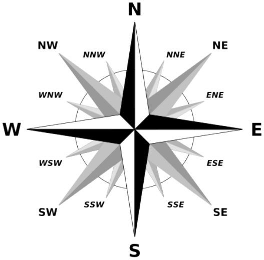


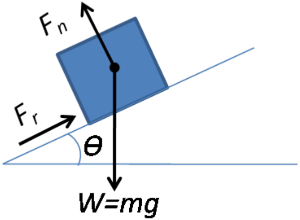
Simplification

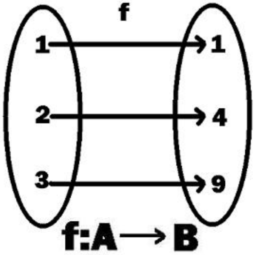
```
y = f(g x)
```

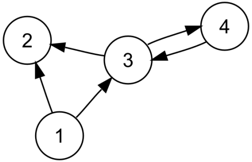
$$\begin{tikzcd}
A & B \arrow[l, "f"'] & C \arrow[l, "g"']
\end{tikzcd}$$

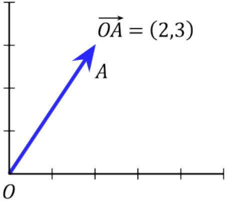
Check the pictures...



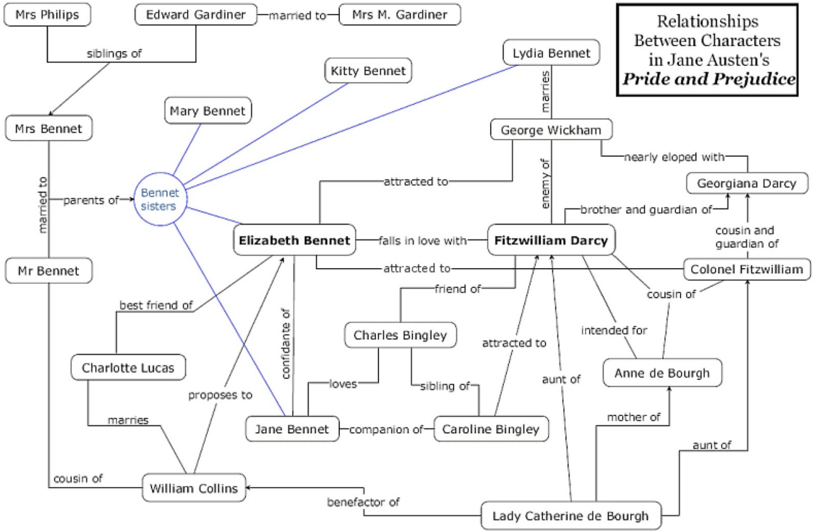








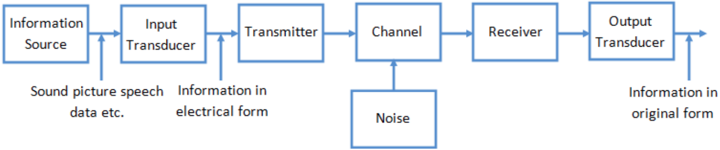
Arrows



Relationships
Between Characters
in Jane Austen's
Pride and Prejudice

Mrs Philips
Edward Gardiner
married to
Mrs M. Gardiner
siblings of
Lydia Bennet
Kitty Bennet
marries
Mary Bennet
Mrs Bennet
George Wickham
nearly eloped with
attracted to
enemy of
Georgiana Darcy
married to
parents of
Bennet sisters
brother and guardian of
cousin and guardian of
Elizabeth Bennet
falls in love with
Fitzwilliam Darcy
Mr Bennet
attracted to
Colonel Fitzwilliam
friend of
cousin of
best friend of
confidante of
Charles Bingley
intended for
attracted to
Charlotte Lucas
Anne de Bourgh
aunt of
loves
sibling of
proposes to
marries
mother of
Jane Bennet
companion of
Caroline Bingley
aunt of
cousin of
William Collins
benefactor of
Lady Catherine de Bourgh

More arrows...



Information Source

Input Transducer

Transmitter

Channel

Receiver

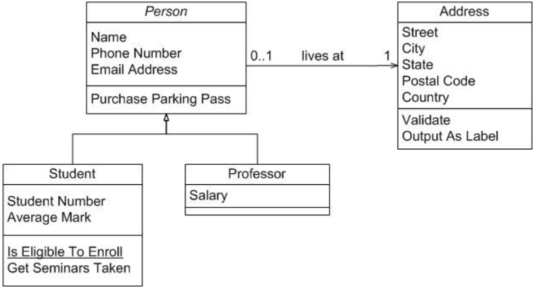
Output Transducer

Noise

Sound picture speech data etc.

Information in electrical form

Information in original form



Person

Name
Phone Number
Email Address

Purchase Parking Pass

0..1 lives at 1

Address

Street
City
State
Postal Code
Country

Validate
Output As Label

Student

Student Number
Average Mark

Is Eligible To Enroll
Get Seminars Taken

Professor

Salary

Composition

$$y = f(g\ x)$$

$$\begin{tikzcd}
A & B \arrow[l, "f"'] & C \arrow[l, "g"']
\end{tikzcd}$$

$$\begin{tikzcd}
A & C \arrow[l, "f \cdot g"']
\end{tikzcd}$$

$$y = (f \cdot g)\ x$$

$$\begin{tikzcd}A & B \arrow[l, "f"'] & C \arrow[l, "g"']\end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "g"] & B \arrow[r, "f"] & A \end{tikzcd}$$

$$\begin{tikzcd}
A & & \\
& B \arrow[ul, "f"'] & \\
& & C \arrow[ul, "g"']
\end{tikzcd}$$

Cf. Unix/Linux pipes

$$\begin{tikzcd} C \arrow[r, "g"] & B \arrow[r, "f"] & A \end{tikzcd}$$

$$g \mid f$$

Composition

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Composition

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

Composition

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

$$f \cdot g \cdot h$$

$$a + b + c$$

Composition

$$store\ c = take\ 10 \cdot \underbrace{nub \cdot (c:)}_{store'\ c}$$

Composition

$$\mathit{store}\ c = \mathit{take}\ 10 \cdot \underbrace{\mathit{nub} \cdot (c:)}_{\mathit{store}'\ c}$$

i.e.

$$\mathit{take}\ 10 \cdot (\mathit{nub} \cdot (c:))$$

Composition

$$store\ c = take\ 10 \cdot \underbrace{nub \cdot (c{:})}_{store'\ c}$$

i.e.

$$take\ 10 \cdot (nub \cdot (c{:}))$$

the same as

$$(take\ 10 \cdot nub) \cdot (c{:})$$

Composition

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

Composition

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

$$a + 0 = 0 + a = a$$

Composition

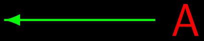
$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

$$a + 0 = 0 + a = a$$

$$f \cdot ? = ? \cdot f = f$$

A


A

$$\begin{tikzcd} A & A \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} A & A \arrow[l, "id"'] \end{tikzcd}$$

$$\begin{tikzcd} A & A \arrow[l, "id"'] \end{tikzcd}$$

$$id\ a = a$$

Identity

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \\
B &
\end{tikzcd}$$

Identity

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \\
B & B
\end{tikzcd}$$

Identity

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \\
B & B \arrow[l, "id"]
\end{tikzcd}$$

Identity

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \arrow[dl, "f"'] \\
B & B \arrow[l, "id"]
\end{tikzcd}$$

Identity

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \arrow[dl, "f"'] \\
B & B \arrow[l, "id"]
\end{tikzcd}$$

$$f \cdot id = f = id \cdot f$$

Composition and identity

Associativity:

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Composition and identity

Associativity:

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

“ Natural-*id*” :

$$f \cdot id = f = id \cdot f$$

$$f \cdot g$$

$$f \cdot g$$

$$f \times g \;?$$

$$f \cdot g$$

$$f \times g \;?$$

$$f + g \;?$$

$$C \xrightarrow{f} B \quad \text{and} \quad A \xrightarrow{g} C$$

$C \xrightarrow{f} B$ and $A \xrightarrow{g} C$

Composition: $A \xrightarrow{f \cdot g} B$

What about

$$\begin{tikzcd}
D \arrow[r, "f"] & B \\
A \arrow[r, "g"] & C
\end{tikzcd}$$

?

Try $D = A$ ...

$$\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Try $D = A$ ...

$$\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$f\ a$ ... $g\ a$

$$\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$(f\ a, g\ a)$

Cartesian product

$$B \times C = \{(b, c) \mid b \in B \wedge c \in C\}$$

Cartesian product

$$B \times C = \{(b, c) \mid b \in B \land c \in C\}$$

$$f\ a \in B$$

Cartesian product

$$B \times C = \{(b, c) \mid b \in B \wedge c \in C\}$$

$$f\ a \in B$$

$$g\ a \in C$$

Cartesian product

$$B \times C = \{(b, c) \mid b \in B \land c \in C\}$$

$$\frac{f\ a \in B \qquad g\ a \in C}{(f\ a, g\ a) \in B \times C}$$

“Split”

$$\begin{tikzcd}
B & B \times C & C \\
& A \arrow[ul, "f"] \arrow[u, "{\langle f, g \rangle}"'] \arrow[ur, "g"']
\end{tikzcd}$$

$$\langle f, g \rangle \; a = (f \; a, g \; a)$$

Product

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Product

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

$$\pi_1 : A \times B \to A$$
$$\pi_1\,(a, b) = a$$

Product

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

$$\pi_1 : A \times B \to A \qquad \pi_2 : A \times B \to B$$

$$\pi_1 (a, b) = a \qquad \pi_2 (a, b) = b$$

Product

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Product

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f,g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\pi_1 \cdot \langle f, g \rangle = f$$

Product

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\pi_1 \cdot \langle f, g \rangle = f$ $\pi_2 \cdot \langle f, g \rangle = g$

Product

$$\langle f, g \rangle$$

Product

$$\langle f, g \rangle$$

f and g in **parallel**

f “split” g

Product

$$\langle f, g \rangle$$

f and g in **parallel**
f “split” g

$$\langle f, g \rangle\ a = (f\ a, g\ a)$$

Cálculo
de
Programas

Class T02 — 26-Sep

Product

$$\begin{tikzcd}
D \\
B \arrow[u, "f"]
\end{tikzcd}$$

Product

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & & C \arrow[u, "g"']
\end{tikzcd}$$

Product

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & C \arrow[u, "g"']
\end{tikzcd}$$

Product

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[ul, "f \cdot \pi_1"'] & C \arrow[u, "g"']
\end{tikzcd}$$

Product

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[ul, "f \cdot \pi_1"'] \arrow[ur, "g \cdot \pi_2"] \arrow[r, "\pi_2"'] & C \arrow[u, "g"']
\end{tikzcd}$$

Product

$$\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "\langle f \cdot \pi_1, g \cdot \pi_2 \rangle" description] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}$$

Product

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "{\langle f \cdot \pi_1, g \cdot \pi_2 \rangle}" description] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}$$

Product

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "f \times g" description] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}$$

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$$

Summing up

$f \cdot g$

Summing up

$f \cdot g$ Sequential composition

Summing up

$f \cdot g$ Sequential composition

$\langle f, g \rangle$

Summing up

$f \cdot g$ Sequential composition

$\langle f, g \rangle$ Parallel composition

Summing up

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition (synchronous)

Summing up

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition (synchronous)
$f \times g$	

Summing up

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition (synchronous)
$f \times g$	Parallel composition

Summing up

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition (synchronous)
$f \times g$	Parallel composition (asynchronous)

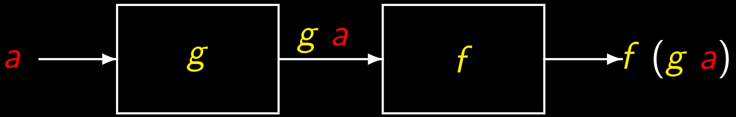
Summing up

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition (synchronous)
$f \times g$	Parallel composition (asynchronous)

Compositional programming

In pictures

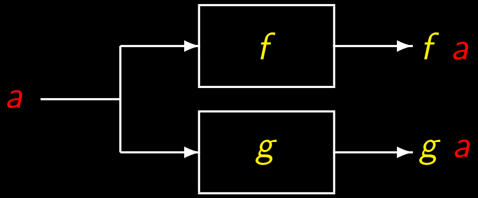
$$(f \cdot g)\ a = f\ (g\ a) \tag{2.6}$$



Function **composition**

In pictures

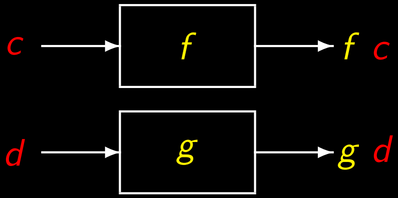
$$\langle f, g \rangle \ a = (f \ a, g \ a) \qquad (2.20)$$



Functional “splits”

In pictures

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle \tag{2.24}$$



Functional **products**

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition (synchronous)
$f \times g$	Parallel composition (asynchronous)

Compositional programming

Problem

Retrieve the address of a civil servant, knowing that she/he can be identified either by a citizen card number (CC) or a fiscal number (NIF).

Problem

Retrieve the address of a civil servant, knowing that she/he can be identified either by a citizen card number (CC) or a fiscal number (NIF).

$$address : Iden \rightarrow Address$$

$$Iden = CC \cup NIF$$

Problem!

$$CC = \mathbb{N}$$

$$NIF = \mathbb{N}$$

Problem!

$$CC = \mathbb{N}$$
$$NIF = \mathbb{N}$$

$$Iden = CC \cup NIF = \mathbb{N} \cup \mathbb{N} = \mathbb{N}$$

Problem!

$$CC = \mathbb{N}$$
$$NIF = \mathbb{N}$$

$$Iden = CC \cup NIF = \mathbb{N} \cup \mathbb{N} = \mathbb{N}$$

$$address : \mathbb{N} \to Address \; (!)$$

In general

We need to fix:

$$m : A \cup B \to C$$

In general

We need to fix:

$$m : A \cup B \to C$$

Let us start from

$$A \cup B = \{\, a \mid a \in A \,\} \cup \{\, b \mid b \in B \,\}$$

Disjoint union

In need of something like...

$$\{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

Disjoint union

In need of something like...

$$\{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

... we define:

$$A + B = \{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

Disjoint union

Clearly,

$$A + B = \{\, i_1\ a \mid a \in A \,\} \cup \{\, i_2\ b \mid b \in B \,\}$$

once we define:

$$i_1\ a = (1, a)$$
$$i_2\ b = (2, b)$$

Disjoint union

Types:

$$i_1 : A \to A + B$$

$$i_2 : B \to A + B$$

Disjoint union

Types:

$$i_1 : A \to A + B$$

$$i_2 : B \to A + B$$

Now think of:

$$m : A + B \to C$$

Disjoint union

Types:

$$i_1 : A \to A + B$$
$$i_2 : B \to A + B$$

Now think of:

$$m : A + B \to C$$

$$\begin{tikzcd}
A + B \arrow[d, "m" description] \\
C
\end{tikzcd}$$

Disjoint union

Types:

$$i_1 : A \to A + B$$

$$i_2 : B \to A + B$$

Now think of:

$$m : A + B \to C$$

$$\begin{tikzcd}
A \arrow[r, "i_1"] & A + B \arrow[d, "m" description] \\
 & C
\end{tikzcd}$$

Disjoint union

Types:

$$i_1 : A \to A + B$$

$$i_2 : B \to A + B$$

Now think of:

$$m : A + B \to C$$

$$\begin{tikzcd}
A \arrow[r, "i_1"] & A + B \arrow[d, "m"] & B \arrow[l, "i_2"'] \\
 & C &
\end{tikzcd}$$

Disjoint union

Types:

$$i_1 : A \to A + B$$
$$i_2 : B \to A + B$$

Now think of:

$$m : A + B \to C$$

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "m"] & B \arrow[l, "i_2"'] \\
 & C &
\end{tikzcd}$$

$$f = m \cdot i_1$$

Disjoint union

Types:

$$i_1 : A \to A + B$$

$$i_2 : B \to A + B$$

Now think of:

$$m : A + B \to C$$

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "m"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

$$f = m \cdot i_1 \qquad g = m \cdot i_2$$

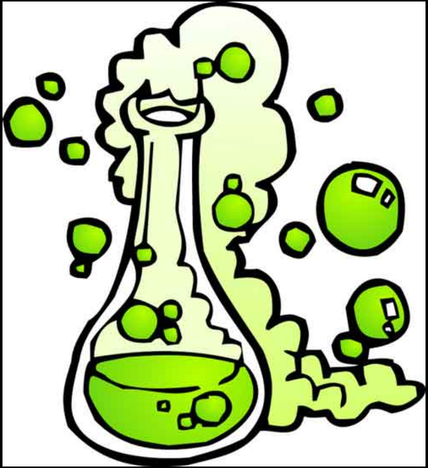
Compare...

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "{[f, g]}"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

Compare...

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Calculus?



$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "{\langle f, g \rangle}" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle" description] \arrow[ur, "g"'] & \\
 & D \arrow[u, "h"] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[u, "\langle f , g \rangle \cdot h" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle" description] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] & \\
 & D \arrow[u, "h"] \arrow[uul, "f \cdot h"] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] & \\
& D \arrow[uul, "f \cdot h"] \arrow[u, "h"] \arrow[uur, "g \cdot h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "f \cdot h"] \arrow[u, "\langle f \cdot h, g \cdot h \rangle" description] \arrow[ur, "g \cdot h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[u, "\langle f, g \rangle \cdot h = \langle f \cdot h, g \cdot h \rangle" description] &
\end{tikzcd}$$

$\times$-Fusion

$$\langle f, g\rangle \cdot h = \langle f \cdot h, g \cdot h\rangle \tag{2.26}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g\rangle"'] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] \arrow[uul, "f \cdot h"] \arrow[uur, "g \cdot h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & & \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \arrow[u, "k"'] \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uul, "h \cdot f"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uul, "h \cdot f"'] \arrow[uur, "k \cdot g"] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "\langle h \cdot f, k \cdot g \rangle" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}$$

$\times$-Absorption

$$(h \times k) \cdot \langle f, g \rangle = \langle h \cdot f, k \cdot g \rangle \tag{2.27}$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
D \arrow[u, "h"] & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & E \arrow[u, "k"] \\
& C \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"'] \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \arrow[u, "k"']
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "h \times k"] & C \arrow[u, "k"']
\end{tikzcd}$$

$$\pi_1 \cdot (h \times k) = h \cdot \pi_1$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \arrow[u, "k"']
\end{tikzcd}$$

$$\pi_1 \cdot (h \times k) = h \cdot \pi_1 \qquad \pi_2 \cdot (h \times k) = k \cdot \pi_2$$

Natural-π_1, natural-π_2

$$\pi_1 \cdot (h \times k) \ = \ h \cdot \pi_1 \tag{2.28}$$

$$\pi_2 \cdot (h \times k) = k \cdot \pi_2 \tag{2.29}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"'] & C \arrow[u, "k"']
\end{tikzcd}$$

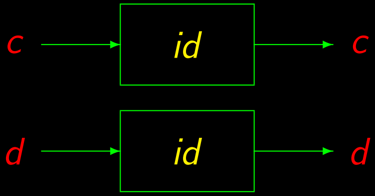
So far...

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition
$f \times g$	Product composition
$[f, g]$	Alternative composition

Compositional programming ☺

Functor-*id*-×

$$id \times id = id \tag{2.31}$$



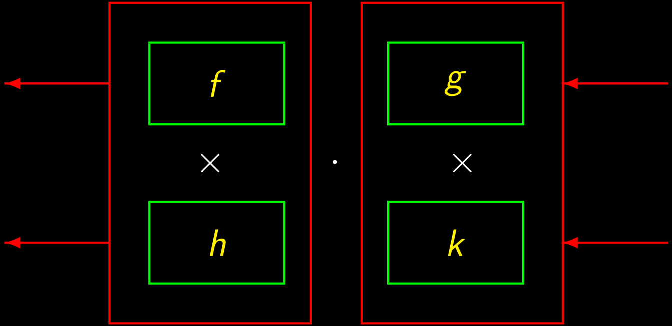
Product of two identities is an identity.

$\times$-Functor

$$(f \times h) \cdot (g \times k) = (f \cdot g) \times (h \cdot k) \tag{2.30}$$

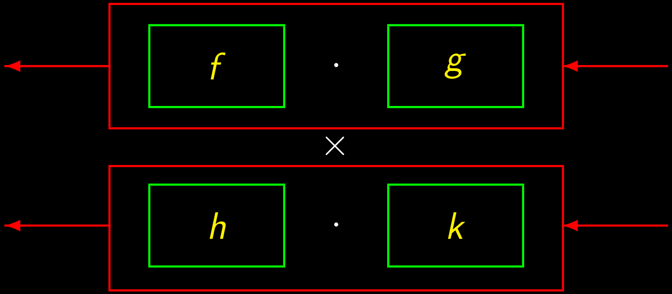
Composition of products is a **product** of compositions.

$\times$-Functor



$$(f \times h) \cdot (g \times k)$$

×-Functor



$$(f \cdot g) \times (h \cdot k)$$

Two basic laws still missing

×-Reflexion $$\langle \pi_1, \pi_2 \rangle = \mathit{id} \qquad (2.32)$$

Two basic laws still missing

$\times$-Reflexion

$$\langle \pi_1, \pi_2 \rangle = id \tag{2.32}$$

$\times$-Eq

$$\langle i, j \rangle = \langle f, g \rangle \;\Leftrightarrow\; \left\{ \begin{array}{l} i = f \\ j = g \end{array} \right. \tag{2.64}$$

Before them, the most important...

Recall ×-**cancellation**:

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "{\langle f, g \rangle}"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\pi_1 \cdot \langle f, g \rangle = f$ $\pi_2 \cdot \langle f, g \rangle = g$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Rightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Leftarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

$\times$-Universal

$$k = \langle f, g \rangle \Leftrightarrow \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

$\times$-Universal

Existence

$$k = \langle f, g \rangle \;\Rightarrow\; \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

"There exists a solution — $k = \langle f, g \rangle$ — for the equations on the right"

$\times$-Universal

Unicity

$$k = \langle f, g \rangle \quad \Leftarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

"Such a solution, $k = \langle f, g \rangle$, is unique"

Equations!

$$\begin{cases} x = 2\,y \\ z = \frac{y}{3} \\ x + y + z = 10 \end{cases}$$

Equations!

$$\begin{cases} x = 2\,y \\ z = \frac{y}{3} \\ x + y + z = 10 \end{cases} \iff \begin{cases} x = 6 \\ z = 1 \\ y = 3 \end{cases}$$

Equations!

Problem

Solve the equation

$$\langle f, g \rangle = id$$

for f *and* g.

Equations!

Calculation

In
$$k = \langle f, g \rangle \Leftrightarrow \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

Equations!

Calculation

In $k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ let $k = id$

Equations!

Calculation

In $k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ let $k = id$

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot id = f \\ \pi_2 \cdot id = g \end{cases}$$

Equations!

Calculation

In
$$k = \langle f, g \rangle \Leftrightarrow \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$
let $k = id$

$$id = \langle f, g \rangle \Leftrightarrow \left\{ \begin{array}{l} \pi_1 = f \\ \pi_2 = g \end{array} \right.$$

Equations!

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Equations!

$$\mathit{id} = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Substituting:

$$\mathit{id} = \langle \pi_1, \pi_2 \rangle$$

Equations!

$$id = \langle f, g \rangle \;\Leftrightarrow\; \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Substituting:

$$id = \langle \pi_1, \pi_2 \rangle$$

×-Reflexion 🙂

Problem

Solve the equation

$$\langle h, k \rangle = \langle f, g \rangle$$

Problem

Solve the equation

$$\langle h, k \rangle = \langle f, g \rangle$$

(1 equation, 4 unknowns)

Calculation

$$\langle h, k \rangle = \langle f, g \rangle$$

Calculation

$$\langle h, k\rangle = \langle f, g\rangle$$

$$\Leftrightarrow \qquad \{\ \times\text{-universal}\ \}$$

$$\begin{cases} \pi_1 \cdot \langle h, k\rangle = f \\ \pi_2 \cdot \langle h, k\rangle = g \end{cases}$$

Calculation

$$\langle h, k \rangle = \langle f, g \rangle$$

$$\Leftrightarrow \quad \{ \times\text{-universal} \}$$

$$\begin{cases} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{cases}$$

$$\Leftrightarrow \quad \{ \times\text{-cancellation} \}$$

$$\begin{cases} h = f \\ k = g \end{cases}$$

Calculation

$$\langle h, k \rangle = \langle f, g \rangle$$

$$\Leftrightarrow \qquad \{ \ \times\text{-universal} \ \}$$

$$\begin{cases} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{cases}$$

$$\Leftrightarrow \qquad \{ \ \times\text{-cancellation} \ \}$$

$$\begin{cases} h = f \\ k = g \end{cases}$$

×-Eq !

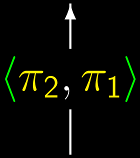
$$\langle \pi_1, \pi_2 \rangle = \mathit{id}$$

$\langle \pi_1, \pi_2 \rangle = \mathit{id}$

$\langle \pi_2, \pi_1 \rangle$?

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?



$\langle \pi_2, \pi_1 \rangle$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & {} \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"]
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & {} & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & B \times A & A \\
& A \times B \arrow[ul, "\pi_2"'] \arrow[u, "\langle \pi_2, \pi_1 \rangle"] \arrow[ur, "\pi_1"] &
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & B \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle" description] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

Problem

Solve

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

for k

Problem

Solve

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

for k

$$\begin{tikzcd}
A \times B \arrow[r, bend left, "\langle \pi_2, \pi_1 \rangle"] & B \times A \arrow[l, bend left, "k"]
\end{tikzcd}$$

Calculation

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

Calculation

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

$$\Leftrightarrow \qquad \{ \times\text{-fusion} \}$$

$$\langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id$$

Calculation

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

$$\Leftrightarrow \quad \{ \ \times\text{-fusion} \ \}$$

$$\langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id$$

$$\Leftrightarrow \quad \{ \ \times\text{-universal} \ \}$$

$$\begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}$$

Calculation

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

$$\Leftrightarrow \quad \{ \ \times\text{-fusion} \ \}$$

$$\langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id$$

$$\Leftrightarrow \quad \{ \ \times\text{-universal} \ \}$$

$$\left\{ \begin{array}{l} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{array} \right.$$

$$\Leftrightarrow \quad \{ \ \text{trivial} \ \}$$

$$\left\{ \begin{array}{l} \pi_1 \cdot k = \pi_2 \\ \pi_2 \cdot k = \pi_1 \end{array} \right.$$

Calculation

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

$$\Leftrightarrow \quad \{ \ \times\text{-fusion} \ \}$$

$$\langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id$$

$$\Leftrightarrow \quad \{ \ \times\text{-universal} \ \}$$

$$\begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}$$

$$\Leftrightarrow \quad \{ \ \text{trivial} \ \}$$

$$\begin{cases} \pi_1 \cdot k = \pi_2 \\ \pi_2 \cdot k = \pi_1 \end{cases}$$

$$\Leftrightarrow \quad \{ \ \times\text{-universal} \ \}$$

$$k = \langle \pi_2, \pi_1 \rangle$$

Swap

$$\begin{tikzcd}
A \times B \arrow[r, bend left, "\langle \pi_2, \pi_1 \rangle"] & B \times A \arrow[l, bend left, "\langle \pi_2, \pi_1 \rangle"]
\end{tikzcd}$$

$$swap = \langle \pi_2, \pi_1 \rangle$$

Swap

$$\begin{tikzcd}A \times B \arrow[r, bend left, "\langle \pi_2, \pi_1 \rangle"] & B \times A \arrow[l, bend left, "\langle \pi_2, \pi_1 \rangle"]\end{tikzcd}$$

$$swap = \langle \pi_2, \pi_1 \rangle$$

$$swap \cdot swap = id$$

So far

$f \cdot g$ Sequential composition

So far

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition

So far

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition

Associativity

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

So far

$f \cdot g$ Sequential composition

$\langle f, g \rangle$ Parallel composition

Associativity

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity?

$$\langle \langle f, g \rangle, h \rangle \overset{?}{=} \langle f, \langle g, h \rangle \rangle$$

So far

$f \cdot g$ Sequential composition

$\langle f, g \rangle$ Parallel composition

Associativity

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity?

$$\langle \langle f, g \rangle, h \rangle \overset{?}{=} \langle f, \langle g, h \rangle \rangle$$

No! but...

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A & \\
& D \arrow[ul, "f"']
\end{tikzcd}$$

$\langle \langle f, g \rangle, h \rangle$

$$\begin{tikzcd}
A & & B \\
& D \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& D \arrow[ul, "f"] \arrow[u, "{\langle f, g\rangle}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & & C \\
& D \arrow[ul, "{\langle f, g\rangle}"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle \langle f, g \rangle, h \rangle$

$$\begin{tikzcd}
A \times B & & C \\
& D \arrow[ul, "{\langle f, g \rangle}"'] \arrow[ur, "h"'] \arrow[u, "{\langle \langle f, g \rangle, h \rangle}" description] &
\end{tikzcd}$$

$\langle \langle f, g \rangle, h \rangle$

$$\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "{\langle f, g \rangle}"] \arrow[u, "{\langle \langle f, g \rangle, h \rangle}" description] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
B & & C \\
& D \arrow[ul, "g"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & D \arrow[ul, "g"] \arrow[u, "\langle g, h \rangle"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
A & & B \times C \\
& D \arrow[ul, "f"] \arrow[ur, "\langle g, h \rangle"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
A & {} & B \times C \\
& D \arrow[ul, "f"'] \arrow[u, "{\langle f, \langle g, h \rangle \rangle}"] \arrow[ur, "{\langle g, h \rangle}"] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
A & A \times (B \times C) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \times C \\
& D \arrow[ul, "f"] \arrow[u, "{\langle f, \langle g, h \rangle \rangle}" description] \arrow[ur, "{\langle g, h \rangle}"'] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "{\langle f, g\rangle}"] \arrow[u, "{\langle\langle f, g\rangle, h\rangle}"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "k"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle\langle f{,}g\rangle{,}h\rangle"] \arrow[ur, "\langle f{,}\langle g{,}h\rangle\rangle"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "k"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle\langle f,g\rangle,h\rangle"] \arrow[ur, "\langle f,\langle g,h\rangle\rangle"'] &
\end{tikzcd}$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \underbrace{\langle\langle f, g\rangle, h\rangle}_{id?} = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \underbrace{\langle\langle f, g\rangle, h\rangle}_{id?} = \langle f, \langle g, h\rangle\rangle$$

Solve $\langle\langle f, g\rangle, h\rangle = id$

$$\langle \langle f, g \rangle, h \rangle = id$$

$$\langle\langle f, g\rangle, h\rangle = id$$

$$\Leftrightarrow \quad \{\ \times\text{-universal}\ \}$$

$$\begin{cases} \pi_1 = \langle f, g\rangle \\ \pi_2 = h \end{cases}$$

$$\langle\langle f, g\rangle, h\rangle = id$$

$$\Leftrightarrow \quad \{ \ \times\text{-universal} \ \}$$

$$\begin{cases} \pi_1 = \langle f, g\rangle \\ \pi_2 = h \end{cases}$$

$$\Leftrightarrow \quad \{ \ \times\text{-universal} \ \}$$

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

Substitute solutions

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

Substitute solutions

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k \cdot \underbrace{\langle \langle f , g \rangle , h \rangle}_{id} = \langle f , \langle g , h \rangle \rangle$$

☺

Substitute solutions

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Slight improvement...

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Slight improvement...

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

$$\Leftrightarrow \quad \{ \ \pi_2 = id \cdot \pi_2 \ \}$$

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle$$

Slight improvement...

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

$$\Leftrightarrow \qquad \{ \ \pi_2 = id \cdot \pi_2 \ \}$$

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle$$

$$\Leftrightarrow \qquad \{ \ f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle \ \}$$

$$k = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle$$

Analogy

In an **arithmetic** context, find k such that

$$k + (x - y) = x + 2$$

holds for any x and y.

Further assume that you **don't know** property:

$$a + b = c \Leftrightarrow a = c - b.$$

How can you find k?

Analogy

Try to cancel $x - y$, ie. solve $x - y = 0$ for x.

You get $x = y$.

Substitute x:

$$k + (y - y) = y + 2$$

You get $k = y + 2$.

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1 , \pi_2 \times id \rangle"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle \langle f , g \rangle , h \rangle"] \arrow[ur, "\langle f , \langle g , h \rangle \rangle"'] &
\end{tikzcd}$$

Back to

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle"] & & A \times (B \times C) \arrow[ll, bend left, "j"] \\
& D \arrow[ul, "\langle \langle f, g \rangle, h \rangle"] \arrow[ur, "\langle f, \langle g, h \rangle \rangle"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "{\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle}"] & A \times (B \times C) \arrow[l, bend left, "{\langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle}"]
\end{tikzcd}$$

$$\begin{tikzcd}(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]\end{tikzcd}$$

$$assocr = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle$$
$$assocl = \langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]
\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]
\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Isomorphism

$$\begin{tikzcd}(A \times B) \times C \arrow[rr, bend left, "assocr"] & \cong & A \times (B \times C) \arrow[ll, bend left, "assocl"]\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Isomorphism

$$\begin{tikzcd}
A \times B \arrow[r, bend left, "swap"] \arrow[r, phantom, "\cong"] & B \times A \arrow[l, bend left, "swap"]
\end{tikzcd}$$

$$swap \cdot swap = id$$

Isomorphism

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, phantom, "\cong"] & & B \arrow[ll, bend left, "g"] \end{tikzcd}$$

$$f \cdot g = id$$

$$g \cdot f = id$$

Isomorphism

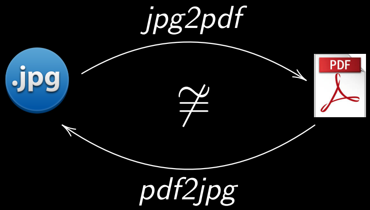
$$\underbrace{\textit{iso}}_{\textit{the same}}\ (\iota\sigma o)$$

Isomorphism

$$\underbrace{\textit{iso}}_{\textit{the same}}\ (\iota\sigma o) + \underbrace{\textit{morphism}}_{\textit{shape}}\ \ (\mu o\rho\phi\iota\sigma\mu o\zeta)$$

“Similar shape”

Practical problem!



jpg2pdf

$\ncong$

pdf2jpg

Practical problem!

$$\begin{tikzcd}
\text{.jpg} \arrow[r, bend left, "jpg2pdf"] \arrow[r, phantom, "\ncong"] & \text{PDF} \arrow[l, bend left, "pdf2jpg"]
\end{tikzcd}$$

$$jpg2pdf \cdot pdf2jpg \neq id$$

$$pdf2jpg \cdot jpg2pdf \neq id$$

Cálculo de Programas

Class T03

Format conversion

Need

$$\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}$$

Format conversion

Need

$$\begin{tikzcd}A \arrow[d, "f"'] \\ B\end{tikzcd}$$

Reusable

$$\begin{tikzcd}C \arrow[d, "r"] \\ D\end{tikzcd}$$

Format conversion

$$\begin{tikzcd}
\text{Need} & \text{Reusable} \\
A \arrow[r, "export"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D
\end{tikzcd}$$

Format conversion

$$\begin{tikzcd}
\text{Need} & \text{Reusable} \\
A \arrow[r, "export"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D \arrow[l, "import"]
\end{tikzcd}$$

Format conversion

$$\begin{tikzcd}
\text{Need} & \text{Reusable} \\
A \arrow[r, "export"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D \arrow[l, "import"]
\end{tikzcd}$$

$$f = import \cdot r \cdot export$$

Format conversion

$$\begin{tikzcd}
\text{Need} & \text{Reusable} \\
A \arrow[r, "export"] \arrow[d, "f"'] \arrow[dr, phantom, "(\cong)"] & C \arrow[d, "r"] \\
B & D \arrow[l, "import"]
\end{tikzcd}$$

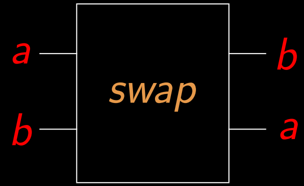
$$f = import \cdot r \cdot export$$

By the way — *swap*

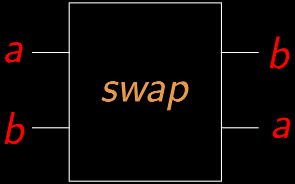
$$\begin{tikzcd} A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"] \end{tikzcd}$$

Isomorphisms are **reversible** computations

By the way — *swap*

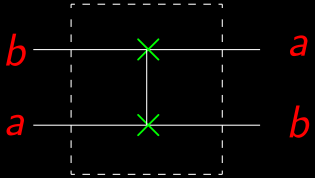


By the way — *swap*



swap is a basic gate in **quantum programming**.

By the way — *swap*



What about alternation $[f, g]$?...

$$[f, g] : A + B \to C$$

$$[f, g]\ x \ = \ \begin{cases} x = i_1\ a \Rightarrow f\ a \\ x = i_2\ b \Rightarrow g\ b \end{cases}$$

Recall...

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "{[f, g]}"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

Compare...

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$+$-Universal

$$k = [f, g] \Leftrightarrow \left\{ \begin{array}{l} k \cdot i_1 = f \\ k \cdot i_2 = g \end{array} \right.$$

$+$-Universal

$$k = [f, g] \Leftrightarrow \left\{ \begin{array}{l} k \cdot i_1 = f \\ k \cdot i_2 = g \end{array} \right.$$

Compare with

$$k = \langle f, g \rangle \Leftrightarrow \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

From $[f, g]$ to $f + g$

$$[f, g] : A + B \to C$$

$$[f, g]\ x \;=\; \begin{cases} x = i_1\ a \Rightarrow f\ a \\ x = i_2\ b \Rightarrow g\ b \end{cases}$$

$f + g$?

Sum of two functions

$$\begin{tikzcd}
D \arrow[d, "f"'] \\
\mathsf{B}
\end{tikzcd}$$

Sum of two functions

$$\begin{tikzcd}
D \arrow[d, "f"'] & & E \arrow[d, "g"] \\
B & & C
\end{tikzcd}$$

Sum of two functions

$$\begin{tikzcd}
D \arrow[d, "f"'] & & E \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_2"]
\end{tikzcd}$$

Sum of two functions

$$\begin{tikzcd}
D \arrow[d, "f"'] \arrow[dr] & & E \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & B+C & C \arrow[l, "i_2"]
\end{tikzcd}$$

Sum of two functions

$$\begin{tikzcd}
D \arrow[d, "f"'] \arrow[dr] & & E \arrow[dl] \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_2"]
\end{tikzcd}$$

Sum of two functions

$$\begin{tikzcd}
D \arrow[r, "i_1"] \arrow[d, "f"'] \arrow[dr] & D + E \arrow[d, "f + g"] & E \arrow[l, "i_2"'] \arrow[d, "g"] \arrow[dl] \\
B \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_2"]
\end{tikzcd}$$

$$f + g = [i_1 \cdot f\,,\, i_2 \cdot g]$$

Coproduct laws

“Just reverse the arrows”, cf.

$+$-Absorption $$[h, k] \cdot (f + g) = [h \cdot f, k \cdot g] \qquad (2.43)$$

Coproduct laws

“Just reverse the arrows”, cf.

+-Absorption $$[h, k] \cdot (f + g) = [h \cdot f, k \cdot g] \tag{2.43}$$

+-Fusion $$f \cdot [h, k] = [f \cdot h, f \cdot k] \tag{2.42}$$

Coproduct laws

“Just reverse the arrows”, cf.

+-Absorption $$[h, k] \cdot (f + g) = [h \cdot f, k \cdot g] \qquad (2.43)$$

+-Fusion $$f \cdot [h, k] = [f \cdot h, f \cdot k] \qquad (2.42)$$

+-Reflexion $$[i_1, i_2] = id \qquad (2.41)$$

Coproduct laws

$+$-Equality
$$[h, k] = [f, g] \Leftrightarrow \begin{cases} h = f \\ k = g \end{cases} \quad (2.66)$$

Coproduct laws

+-Equality
$$[h, k] = [f, g] \Leftrightarrow \left\{ \begin{array}{l} h = f \\ k = g \end{array} \right. \tag{2.66}$$

+-Functor
$$(h + k) \cdot (f + g) = h \cdot f + k \cdot g \tag{2.44}$$

Coproduct laws

$+$-Equality
$$[h, k] = [f, g] \Leftrightarrow \begin{cases} h = f \\ k = g \end{cases} \tag{2.66}$$

$+$-Functor
$$(h + k) \cdot (f + g) = h \cdot f + k \cdot g \tag{2.44}$$

$+$-Functor-*id*
$$id + id = id \tag{2.45}$$

and so on

All these laws can be found in the reference sheet

(WWW → **Material**)

Summary

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition
$f \times g$	Product composition

Summary

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition
$f \times g$	Product composition
$[f, g]$	Alternative composition

Summary

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition
$f \times g$	Product composition
$[f, g]$	Alternative composition
$f + g$	Structural alternation (coproduct)

Summary

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition
$f \times g$	Product composition
$[f, g]$	Alternative composition
$f + g$	Structural alternation (coproduct)

Compositional programming ☺

TBC...