# Cálculo de Programas
## *Algebra of Programming*

UNIVERSIDADE DO MINHO
Lic. Ciências da Computação (3º ano)
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2025/26 - Ficha *( Exercise sheet )* nr. 11

1. O diagrama que se segue

   *The following diagram*

$$
\begin{tikzcd}
\mathsf{T}_1 \arrow[d, "\llparenthesis\, \mathsf{in}_2 \cdot \alpha\, \rrparenthesis"'] & & \mathsf{F}\ \mathsf{T}_1 \arrow[ll, "\mathsf{in}_1"'] \arrow[d, "\mathsf{F}\ \llparenthesis\, \mathsf{in}_2 \cdot \alpha\, \rrparenthesis"] \\
\mathsf{T}_2 \arrow[d, "\llparenthesis\, g\, \rrparenthesis"'] & \mathsf{G}\ \mathsf{T}_2 \arrow[l, "\mathsf{in}_2"] \arrow[d, "\mathsf{G}\ \llparenthesis\, g\, \rrparenthesis"] & \mathsf{F}\ \mathsf{T}_2 \arrow[l, "\alpha"'] \arrow[d, "\mathsf{F} \llparenthesis\, g\, \rrparenthesis"] \\
C & \mathsf{G}\ C \arrow[l, "g"] & \mathsf{F} C \arrow[l, "\alpha"]
\end{tikzcd}
\tag{F1}
$$

compõe dois catamorfismos envolvendo os tipos $\mathsf{T}_1$ ($\mathsf{F}$-recursivo) e $\mathsf{T}_2$ ($\mathsf{G}$-recursivo):

*involves the types $\mathsf{T}_1$ ($\mathsf{F}$-recursive) e $\mathsf{T}_2$ ($\mathsf{G}$-recursive) and two catamorphims:*

$$
\begin{array}{l}
\llparenthesis\, g\, \rrparenthesis : \mathsf{T}_2 \to C \\
\llparenthesis\, \mathsf{in}_2 \cdot \alpha\, \rrparenthesis : \mathsf{T}_1 \to \mathsf{T}_2
\end{array}
$$

Considere-se o caso:

*Consider the special case that follows:*

$$
\begin{array}{l}
\mathsf{T}_1 = \mathsf{T}_2 = \mathsf{LTree}\ A \\
\mathsf{F}\ f = id + f \times f \\
\alpha = id + \mathsf{swap} \\
\mathsf{in}_2 = [Leaf\ , Fork]
\end{array}
$$

Desenvolver

*Unfold*

$$
mirror = \llparenthesis\, \mathsf{in}_2 \cdot \alpha\, \rrparenthesis \tag{F2}
$$

até se obter uma definição completamente *pointwise*.

*until reaching a fully pointwise definition.*

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2. A lei

   *Law*

$$
\llparenthesis\, g\, \rrparenthesis \cdot \llparenthesis\, \mathsf{in}_2 \cdot \alpha\, \rrparenthesis = \llparenthesis\, g \cdot \alpha\, \rrparenthesis \quad \Leftarrow \quad \mathsf{G}\ f \cdot \alpha = \alpha \cdot \mathsf{F}\ f \tag{F3}
$$

verifica-se — cf. diagrama (F1) — onde a condição

*holds — see diagram (F1) — where condition*

$$\mathsf{G}\, f \cdot \alpha = \alpha \cdot \mathsf{F}\, f \tag{F4}$$

mais não é que a propriedade grátis de $\alpha : \mathsf{F}\ X \to \mathsf{G}\ X$. Apresente justificações para a seguinte demonstração de (F3):

*is nothing more than the free property of $\alpha : \mathsf{F}\ X \to \mathsf{G}\ X$. Provide justifications for the following proof of (F3):*

$$
\begin{array}{ll}
& (\!| g |\!) \cdot (\!| \mathsf{in}_2 \cdot \alpha |\!) = (\!| g \cdot \alpha |\!) \\
\Leftarrow & \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& (\!| g |\!) \cdot \mathsf{in}_2 \cdot \alpha = g \cdot \alpha \cdot \mathsf{F}\ (\!| g |\!) \\
\equiv & \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& g \cdot \mathsf{G}\ (\!| g |\!) \cdot \alpha = g \cdot \alpha \cdot \mathsf{F}\ (\!| g |\!) \\
\Leftarrow & \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& \mathsf{G}\ (\!| g |\!) \cdot \alpha = \alpha \cdot \mathsf{F}\ (\!| g |\!) \\
\Leftarrow & \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& \mathsf{G}\, f \cdot \alpha = \alpha \cdot \mathsf{F}\, f
\end{array}
$$

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3. Na sequência da questão 1 acima, suponha que se tem $(\!| g |\!) = \mathit{mirror}$ em (F1). Mostre por (F3) que

*As follow up of question 1 above, suppose that one has $(\!| g |\!) = \mathit{mirror}$ in (F1). Show by (F3) that*

$$\mathit{mirror} \cdot \mathit{mirror} = \mathit{id} \tag{F5}$$

se verifica.

*holds.*

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4. Recordando a definição $\mathsf{T}\, f = (\!| \mathsf{in} \cdot \mathsf{B}\ (f, id) |\!)$, mostre que a lei de absorção-cata,

*Recalling definition $\mathsf{T}\ f = (\!| \mathsf{in} \cdot \mathsf{B}\ (f, id) |\!)$, show that the law of absorption-cata,*

$$(\!| g |\!) \cdot \mathsf{T}\, f \quad = \quad (\!| g \cdot \mathsf{B}\ (f, id) |\!)$$

é um caso particular de (F3).

*is a special case of (F3).*

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5. Todo o ciclo-*while* que termina pode ser definido por

*Every terminating* while-*loop can be defined by*

$$\mathbf{while}\ p\ f\ g = \mathbf{tailr}\ ((g + f) \cdot (\neg \cdot p)?) \tag{F6}$$

recorrendo ao combinador de *"tail recursion"*

*using the* "tail recursion" *combinator*

$$\mathbf{tailr}\ f = [\![ \mathsf{join}, f ]\!] \tag{F7}$$

que é um hilomorfismo de base $\mathsf{B}\ (X, Y) = X + Y$, para $\mathsf{join} = [id\ , id]$.

*which is a hylomorphism of basis $\mathsf{B}\ (X, Y) = X + Y$, for $\mathsf{join} = [id\ , id]$.*

Derive a definição *pointwise* de $\mathbf{while}\ p\ f\ g$, sabendo que qualquer $h = [\![ f, g ]\!]$ que termina é tal que $h = f \cdot \mathsf{F}\ h \cdot g$.

*Derive the* pointwise *definition of $\mathbf{while}\ p\ f\ g$, knowing that any terminating $h = [\![ f, g ]\!]$ is such that $h = f \cdot \mathsf{F}\ h \cdot g$.*

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6. Considere a seguinte lei de fusão de $\mathbf{tailr}$, válida sempre que $(\mathbf{tailr}\ g) \cdot f$ termina:

*Consider the following fusion-law of* $\mathbf{tailr}$*, valid whenever* $(\mathbf{tailr}\ g) \cdot f$ *terminates:*

$$(\mathbf{tailr}\ g) \cdot f = \mathbf{tailr}\ h \ \Leftarrow\ (id + f) \cdot h = g \cdot f \tag{F8}$$

Complete a seguinte demonstração dessa lei.

*Complete de following proof of (F8).*

$$
\begin{array}{ll}
 & (\mathbf{tailr}\ g) \cdot f = \mathbf{tailr}\ h \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & (\!| \nabla |\!) \cdot [\![(g)]\!] \cdot f = (\!| \nabla |\!) \cdot [\![(h)]\!] \\
\Leftarrow & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & [\![(g)]\!] \cdot f = [\![(h)]\!] \\
\Leftarrow & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & g \cdot f = (id + f) \cdot h \\
\Box &
\end{array}
$$

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7. Um mónade é um functor $\mathsf{T}$ equipado com duas funções $\mu$ e $u$,

*A monad is a functor* $\mathsf{T}$ *equipped with two functions* $\mu$ *and* $u$

$$A \xrightarrow{\ u\ } \mathsf{T}\ A \xleftarrow{\ \mu\ } \mathsf{T}\ (\mathsf{T}\ A)$$

que satisfazem determinadas propriedades (ver formulário). Partindo da definição

*satisfying a few properties (see formula sheet). Starting from the definition of monadic composition,*

$$f \bullet g = \mu \cdot \mathsf{T}\ f \cdot g \tag{F9}$$

de *composição monádica*, demonstre a seguinte propriedade sua:

*prove the following property:*

$$f \bullet [g\ , h] = [f \bullet g\ , f \bullet h] \tag{F10}$$