Cálculo de Programas
Algebra of Programming

UNIVERSIDADE DO MINHO
Lic. em Engenharia Informática (3º ano)
Lic. Ciências da Computação (2º ano)

2024/25 - Ficha *(Exercise sheet)* nr. 11

1. O diagrama que se segue

   The following diagram

$$
\begin{tikzcd}
\mathsf{T}_1 \arrow[d, "\llparenthesis\, \mathsf{in}_2 \cdot \alpha \,\rrparenthesis"'] & & \mathsf{F}\ \mathsf{T}_1 \arrow[ll, "\mathsf{in}_1"'] \arrow[d, "\mathsf{F}\ \llparenthesis\, \mathsf{in}_2 \cdot \alpha \,\rrparenthesis"] \\
\mathsf{T}_2 \arrow[d, "\llparenthesis\, g \,\rrparenthesis"'] & \mathsf{G}\ \mathsf{T}_2 \arrow[l, "\mathsf{in}_2"] \arrow[d, "\mathsf{G}\ \llparenthesis\, g \,\rrparenthesis"] & \mathsf{F}\ \mathsf{T}_2 \arrow[l, "\alpha"'] \arrow[d, "\mathsf{F}\llparenthesis\, g \,\rrparenthesis"] \\
C & \mathsf{G}\ C \arrow[l, "g"] & \mathsf{F}C \arrow[l, "\alpha"]
\end{tikzcd}
\tag{F1}
$$

   compõe dois catamorfismos envolvendo dois, tipos T_1 (F-recursivo) e T_2 (G-recursivo):

   involves the types T_1 (F-recursive) e T_2 (G-recursive) and two catamorphims:

$$
\begin{aligned}
&\llparenthesis\, g \,\rrparenthesis : \mathsf{T}_2 \to C \\
&\llparenthesis\, \mathsf{in}_2 \cdot \alpha \,\rrparenthesis : \mathsf{T}_1 \to \mathsf{T}_2
\end{aligned}
$$

   Considere-se o caso:

   Consider the special case that follows:

$$
\begin{aligned}
&\mathsf{T}_1 = \mathsf{T}_2 = \mathsf{LTree}\ A \\
&\mathsf{F}\ f = id + f \times f \\
&\alpha = id + \mathsf{swap} \\
&\mathsf{in}_2 = [\mathit{Leaf}\ , \mathit{Fork}]
\end{aligned}
$$

   Desenvolver

   Unfold

$$
\mathit{mirror} = \llparenthesis\, \mathsf{in}_2 \cdot \alpha \,\rrparenthesis \tag{F2}
$$

   até se obter uma definição completamente *pointwise*.

   until reaching a fully pointwise definition.

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2. A lei

   Law

$$
\llparenthesis\, g \,\rrparenthesis \cdot \llparenthesis\, \mathsf{in}_2 \cdot \alpha \,\rrparenthesis = \llparenthesis\, g \cdot \alpha \,\rrparenthesis \quad \Leftarrow \quad \mathsf{G}\ f \cdot \alpha = \alpha \cdot \mathsf{F}\ f \tag{F3}
$$

verifica-se — cf. diagrama (F1) — onde a condição

holds — see diagram (F1) — where condition

$$\mathsf{G}\,f \cdot \alpha = \alpha \cdot \mathsf{F}\,f \tag{F4}$$

mais não é que a propriedade grátis de $\alpha : \mathsf{F}\ X \to \mathsf{G}\ X$. Apresente justificações para a seguinte demonstração de (F3):

is nothing more than the free property of $\alpha : \mathsf{F}\ X \to \mathsf{G}\ X$. Provide justifications for the following proof of (F3):

$$
\begin{array}{ll}
& (\!|\, g \,|\!) \cdot (\!|\, \mathsf{in}_2 \cdot \alpha \,|\!) = (\!|\, g \cdot \alpha \,|\!) \\
\Leftarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& (\!|\, g \,|\!) \cdot \mathsf{in}_2 \cdot \alpha = g \cdot \alpha \cdot \mathsf{F}\ (\!|\, g \,|\!) \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& g \cdot \mathsf{G}\ (\!|\, g \,|\!) \cdot \alpha = g \cdot \alpha \cdot \mathsf{F}\ (\!|\, g \,|\!) \\
\Leftarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \mathsf{G}\ (\!|\, g \,|\!) \cdot \alpha = \alpha \cdot \mathsf{F}\ (\!|\, g \,|\!) \\
\Leftarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \mathsf{G}\,f \cdot \alpha = \alpha \cdot \mathsf{F}\,f
\end{array}
$$

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3. Na sequência da questão 1 acima, suponha que se tem $(\!|\, g \,|\!) = \mathit{mirror}$ em (F1). Mostre por (F3) que

As follow up of question 1 above, suppose that one has $(\!|\, g \,|\!) = \mathit{mirror}$ in (F1). Show by (F3) that

$$\mathit{mirror} \cdot \mathit{mirror} = \mathit{id} \tag{F5}$$

se verifica.

holds.

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4. Recordando a definição $\mathsf{T}\,f = (\!|\, \mathsf{in} \cdot \mathsf{B}\ (f, \mathit{id}) \,|\!)$, mostre que a lei de absorção-cata,

Recalling definition $\mathsf{T}\ f = (\!|\, \mathsf{in} \cdot \mathsf{B}\ (f, \mathit{id}) \,|\!)$, show that the law of absorption-cata,

$$(\!|\, g \,|\!) \cdot \mathsf{T}\,f \quad = \quad (\!|\, g \cdot \mathsf{B}\ (f, \mathit{id}) \,|\!)$$

é um caso particular de (F3).

is a special case of (F3).

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5. Todo o ciclo-*while* que termina pode ser definido por

Every terminating while-*loop can be defined by*

$$\mathbf{while}\ p\ f\ g = \mathbf{tailr}\ ((g + f) \cdot (\neg \cdot p)?) \tag{F6}$$

recorrendo ao combinador de *"tail recursion"*

using the "tail recursion" *combinator*

$$\mathbf{tailr}\ f = [\![\mathsf{join}, f]\!] \tag{F7}$$

que é um hilomorfismo de base $\mathsf{B}\ (X, Y) = X + Y$, para $\mathsf{join} = [\mathit{id}\ , \mathit{id}]$.

which is a hylomorphism of basis $\mathsf{B}\ (X, Y) = X + Y$, for $\mathsf{join} = [\mathit{id}\ , \mathit{id}]$.

Derive a definição *pointwise* de $\mathbf{while}\ p\ f\ g$, sabendo que qualquer $h = [\![f, g]\!]$ que termina é tal que $h = f \cdot \mathsf{F}\ h \cdot g$.

Derive the pointwise *definition of $\mathbf{while}\ p\ f\ g$, knowing that any terminating $h = [\![f, g]\!]$ is such that $h = f \cdot \mathsf{F}\ h \cdot g$.*

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6. Considere a seguinte lei de fusão de **tailr**, válida sempre que $(\mathbf{tailr}\ g) \cdot f$ termina:

Consider the following fusion-law of **tailr**, *valid whenever* $(\mathbf{tailr}\ g) \cdot f$ *terminates:*

$$(\mathbf{tailr}\ g) \cdot f = \mathbf{tailr}\ h \ \Leftarrow\ (id + f) \cdot h = g \cdot f \qquad \text{(F8)}$$

Complete a seguinte demonstração dessa lei.

Complete de following proof of (F8).

$$
\begin{array}{cl}
 & (\mathbf{tailr}\ g) \cdot f = \mathbf{tailr}\ h \\
\equiv & \quad \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \ \} \\
 & (\!|\, \nabla \,|\!) \cdot [\!(g)\!] \cdot f = (\!|\, \nabla \,|\!) \cdot [\!(h)\!] \\
\Leftarrow & \quad \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \ \} \\
 & [\!(g)\!] \cdot f = [\!(h)\!] \\
\Leftarrow & \quad \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \ \} \\
 & g \cdot f = (id + f) \cdot h \\
\Box &
\end{array}
$$

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7. Um mónade é um functor T equipado com duas funções μ e u,

A monad is a functor T *equipped with two functions* μ *and* u

$$A \xrightarrow{u} \mathsf{T}\ A \xleftarrow{\mu} \mathsf{T}\ (\mathsf{T}\ A) \qquad \text{(F9)}$$

que satisfazem as propriedades

satisfying

$$\mu \cdot u = id = \mu \cdot \mathsf{T}\ u \qquad \text{(F10)}$$

$$\mu \cdot \mu = \mu \cdot \mathsf{T}\ \mu \qquad \text{(F11)}$$

para além das respectivas propriedades "grátis", onde $\mathsf{T}^2\ f$ abrevia $\mathsf{T}\ (\mathsf{T}\ f)$:

in addition to their "free" properties, where $\mathsf{T}^2\ f$ *abbreviates* $\mathsf{T}\ (\mathsf{T}\ f)$*:*

$$\mathsf{T}\ f \cdot u = u \cdot f \qquad \text{(F12)}$$

$$\mathsf{T}\ f \cdot \mu = \mu \cdot \mathsf{T}^2\ f \qquad \text{(F13)}$$

Partindo da definição

Starting from the definition of monadic composition,

$$f \bullet g = \mu \cdot \mathsf{T}\ f \cdot g \qquad \text{(F14)}$$

de *composição monádica*, demonstre os factos seguintes:

prove the following facts:

$$
\begin{array}{rcll}
\mu & = & id \bullet id & \text{(F15)} \\
f \bullet u = f & \wedge & f = u \bullet f & \text{(F16)} \\
(f \cdot g) \bullet h & = & f \bullet (\mathsf{T}\ g \cdot h) & \text{(F17)} \\
\mathsf{T}\ f & = & (u \cdot f) \bullet id & \text{(F18)}
\end{array}
$$