Cálculo de Programas
Algebra of Programming

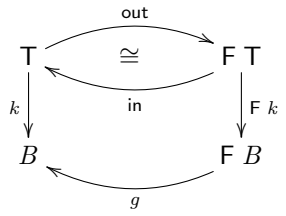
UNIVERSIDADE DO MINHO
Lic. em Engenharia Informática (3º ano)
Lic. Ciências da Computação (2º ano)

2024/25 - Ficha *(Exercise sheet)* nr. 7

O quadro abaixo representa a **propriedade universal** que define o combinador **catamorfismo**, com duas instâncias — números naturais $\mathbb{N}_0$ e listas finitas A^*, onde $\widehat{f}$ abrevia $\mathsf{uncurry}\, f$.

*The table below depicts the **universal property** that defines the **catamorphism** combinator, with two instances — natural numbers $\mathbb{N}_0$ and finite lists A^*, where $\widehat{f}$ abbreviates $\mathsf{uncurry}\, f$:*

Catamorfismo *(Catamorphism)*:



$$k = (\!|\, g \,|\!) \;\Leftrightarrow\; k \cdot \mathsf{in} = g \cdot \mathsf{F}\; k \qquad \text{(F1)}$$

Listas *(Lists)*:

$$\begin{cases} \mathsf{T} = A^* \\ \begin{cases} \mathsf{in} = [\mathsf{nil}\,, \mathsf{cons}] \\ \mathsf{nil}\ \_ = [\,] \\ \mathsf{cons}\ (h, t) = h : t \end{cases} \\ \begin{cases} \mathsf{F}\ X = 1 + A \times X \\ \mathsf{F}\ f = id + id \times f \end{cases} \end{cases} \qquad \mathsf{foldr}\ f\ i = (\!|\, [\underline{i}\,, \widehat{f}] \,|\!)$$

Números naturais *(Natural numbers)*:

$$\begin{cases} \mathsf{T} = \mathbb{N}_0 \\ \begin{cases} \mathsf{in}_{\mathbb{N}_0} = [\underline{0}\,, \mathsf{succ}] \\ \mathsf{succ}\ n = n + 1 \end{cases} \\ \begin{cases} \mathsf{F}\ X = 1 + X \\ \mathsf{F}\ f = id + f \end{cases} \end{cases} \qquad \mathsf{for}\ b\ i = (\!|\, [\underline{i}\,, b] \,|\!)$$

1. Fazendo $\mathsf{T} = \mathbb{N}_0$, codifique — recorrendo à biblioteca Cp.hs e à definição de out feita numa ficha anterior — o combinador:

   Taking $\mathsf{T} = \mathbb{N}_0$, encode — loading the Cp.hs library and using out *defined in a previous exercise sheet, the combinator:*

   $$(\!|\, g \,|\!) = g \cdot (id + (\!|\, g \,|\!)) \cdot \mathsf{out} \qquad \text{(F2)}$$

   De seguida implemente e teste a seguinte função:

   Then implement and test de following function:

   $$rep\ a = (\!|\, [\mathsf{nil}\,, (a:)] \,|\!) \qquad \text{(F3)}$$

O que faz ela? *What is its purpose?*

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2. Identifique como catamorfismos de listas ($k = (\!|\, g \,|\!)$) as funções seguintes, indicando o gene g para cada caso (apoie a sua resolução com diagramas):

   Identify as list catamorphisms ($k = (\!|\, g \,|\!)$) the following functions, indicating the corresponding 'gene' g for each case (support your answer with diagrams):

   (a) k é a função que multiplica todos os elementos de uma lista.

   (a) k is the function that multiplies all elements of a list.

   (b) $k = \mathsf{reverse}$

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   (c) $k = \mathsf{concat}$

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   (d) k é a função $\mathsf{map}\ f$, para um dado $f : A \to B$.

   (d) k is the function $\mathsf{map}\ f$, for a given $f : A \to B$.

   (e) k é a função que calcula o máximo de uma lista de números naturais ($\mathbb{N}_0{}^*$).

   (e) k is the function that calculates the maximum of a list of natural numbers ($\mathbb{N}_0{}^*$).

   (f) $k = \mathsf{filter}\ p$ onde:

   (f) $k = \mathsf{filter}\ p$ where:

$$\mathsf{filter}\ p\ [\,] = [\,]$$
$$\mathsf{filter}\ p\ (h : t) = x +\!\!+ \mathsf{filter}\ p\ t\ \mathbf{where}\ x = \mathbf{if}\ (p\ h)\ \mathbf{then}\ [h]\ \mathbf{else}\ [\,]$$

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3. A função seguinte, em Haskell

   The following function, in Haskell

$$sumprod\ a\ [\,] = 0$$
$$sumprod\ a\ (h : t) = a * h + sumprod\ a\ t$$

é o catamorfismo de listas

is the list-catamorphism

$$sumprod\ a \quad = \quad (\!|\, [\mathsf{zero}\ , \mathsf{add} \cdot ((a*) \times id)] \,|\!) \tag{F4}$$

onde $\mathsf{zero} = \underline{0}$ e $\mathsf{add}\ (x, y) = x + y$. Como exemplo de aplicação da propriedade de **fusão-cata** para listas, demonstre a igualdade

where $\mathsf{zero} = \underline{0}$ *and* $\mathsf{add}\ (x, y) = x + y$. *As an example of application of **cata-fusion**, prove the equality*

$$sumprod\ a \quad = \quad (a*) \cdot \mathsf{sum} \tag{F5}$$

onde $\mathsf{sum} = (\!|\, [\mathsf{zero}\ , \mathsf{add}] \,|\!)$. **NB:** não ignore propriedades elementares da aritmética que lhe possam ser úteis.

where $\mathsf{sum} = (\!|\, [\mathsf{zero}\ , \mathsf{add}] \,|\!)$. ***NB:** take into account elementary arithmetic properties that may be useful.*

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4. A função $\mathsf{foldr}\ \overline{\pi_2}\ i$ é necessariamente uma função constante. Qual? Justifique com o respectivo cálculo.

   Function $\mathsf{foldr}\ \overline{\pi_2}\ i$ is a constant function, for any i – which constant function? Write down your calculations.

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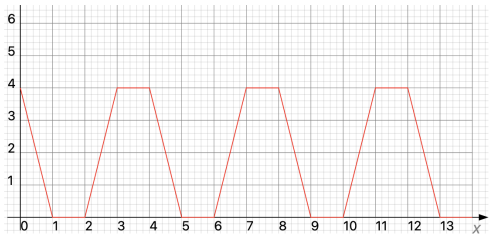
5. A figura representa a função $\pi_1 \cdot aux$, para aux definida ao lado:

   The figure plots $\pi_1 \cdot aux$, for aux defined aside:



$$aux = \mathsf{for}\ loop\ (4,-2)\ \mathbf{where}$$
$$loop\ (a,b) = (2+b, 2-a)$$

Partindo da definição do combinador for $b\ i = (\!|\, [\underline{i}\ , b]\, |\!)$, para $\mathsf{F} = id + f$ e $\mathsf{in} = [\underline{0}\ , \mathsf{succ}]$, resolva em ordem a f e g a equação

Starting from the definition of for $b\ i = (\!|\, [\underline{i}\ , b]\, |\!)$, *for* $\mathsf{F} = id + f$ *and* $\mathsf{in} = [\underline{0}\ , \mathsf{succ}]$, *solve for* f *and* g *the equation*

$$\langle f, g\rangle = aux$$

por aplicação da lei de recursividade mútua, entregando as definições de f e g em notação *pointwise*.

by the mutual recursion law, delivering the definitions of f *and* g *in* pointwise *notation.*

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6. Mostre que a lei da recursividade mútua generaliza a mais do que duas funções, neste caso três:

Show that the mutual recursion law generalizes to more than two functions (three, in the following case):

$$\left\{\begin{array}{l} f\cdot \mathsf{in} = h\cdot \mathsf{F}\ \langle\langle f, g\rangle, j\rangle \\ g\cdot \mathsf{in} = k\cdot \mathsf{F}\ \langle\langle f, g\rangle, j\rangle \\ j\cdot \mathsf{in} = l\cdot \mathsf{F}\ \langle\langle f, g\rangle, j\rangle \end{array}\right. \quad \equiv \quad \langle\langle f, g\rangle, j\rangle = (\!|\, \langle\langle h, k\rangle, l\rangle\, |\!) \tag{F6}$$

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7. Considere o functor

Consider functor

$$\left\{\begin{array}{l} \mathsf{T}\ X = X\times X \\ \mathsf{T}\ f = f\times f \end{array}\right. \tag{F7}$$

e as funções

and functions

$$\left\{\begin{array}{l} \mu = \pi_1\times\pi_2 \\ u = \langle id, id\rangle \end{array}\right. \tag{F8}$$

(a) Mostre que T é de facto um functor:

(a) Show that T *is indeed a functor:*

$$\mathsf{T}\ id = id \tag{F9}$$
$$\mathsf{T}\ (f\cdot g) = \mathsf{T}\ f\cdot \mathsf{T}\ g \tag{F10}$$

(b) Demonstre a propriedade:

(b) Prove the following property:

$$\mu\cdot \mathsf{T}\ u = id = \mu\cdot u$$

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8. **Questão prática** — Este problema não irá ser abordado em sala de aula. Os alunos devem tentar resolvê-lo em casa e, querendo, publicarem a sua solução no canal **#geral** do Slack, com vista à sua discussão com colegas.

Open assignment *— This assignment will not be addressed in class. Students should try to solve it at home and, whishing so, publish their solutions in the* ***#geral*** *Slack channel, so as to trigger discussion among other colleagues.*

***Problem requirements**:*

In the context of a sporting competition (e.g. football league), suppose you have access to the history of all games of the competition, organized by date, in $db_1 :: [(Date, [Game])]$ *(using Haskell syntax). Also given is* $db_2 :: [(Game, [Player])]$ *indicating which palyers played in which game.*

A sport-tv commentator asks you to derive from db_1 *and from* db_2 *the list, ordered by player name, of the dates on which each player played, also ordered. Define, in Haskell, a function* f *implementing such a derivation:*

$$f :: [(Date, [Game])] \to [(Game, [Player])] \to [(Player, [Date])]$$

Challenged by these requirements, ChatGPT gave the solution given below in the black text boxes, which doesn't type but is the sort of solution to be expected.

*In the context of this course, you can write **far less** code to implement* f*!*

Why and how?

```
import Data.List (sort, nub)
type Date = String    -- You can replace String with an appropriate Date type
type Player = String
type Game = String
```

```
-- Helper function to extract unique player names from a list of games
extractPlayers :: [(Game, [Player])] -> [Player]
extractPlayers = nub . concatMap π₂

-- Helper function to map players to the dates they played on
mapPlayersToDates :: [(Date, [Game])] ->
  [(Game, [Player])] -> [(Player, [Date])]
mapPlayersToDates db₁ db₂ = [(player, sort $ nub playedDates)]
  where
    players = extractPlayers db₂
    playedDates player = [date | (date, games) ← db₁,
      any (λ(game, players) → player ∈ players ∧ game ∈ games) db₂]
```

```
-- Main function f
f :: [(Date, [Game])] -> [(Game, [Player])] -> [(Player, [Date])]
f db₁ db₂ = mapPlayersToDates db₁ db₂
```

```
-- Example usage:
main :: IO ()
main = do
  let db₁ = [("2023-10-01", ["Game1", "Game2"]),
    ("2023-10-02", ["Game2", "Game3"])]
  let db₂ = [("Game1", ["PlayerA", "PlayerB"]),
    ("Game2", ["PlayerA", "PlayerC"]),
    ("Game3", ["PlayerB", "PlayerC"])]
  let result = f db₁ db₂
  print result
```

□