Cálculo de Programas
Algebra of Programming

Lic. Ciências da Computação (2º ano)
Lic./Mest.Int. em Engenharia Informática (3º ano)
UNIVERSIDADE DO MINHO

2024/25 - Ficha nr.° 3

1. Considere o diagrama

Consider the following diagram

$$(A \times B) \times C \;\underset{\mathsf{assocl}}{\overset{\mathsf{assocr}}{\cong}}\; A \times (B \times C)$$

onde $\mathsf{assocl} = \langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle$. Apresente justificações para o cálculo que se segue em que se resolve em ordem a assocr a equação $\mathsf{assocl} \cdot \mathsf{assocr} = id$:

where $\mathsf{assocl} = \langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle$. *The reasoning below solves the equation* $\mathsf{assocl} \cdot \mathsf{assocr} = id$ *for variable* assocr. *Fill in justifications for each step in the reasoning:*

$$\mathsf{assocl} \cdot \mathsf{assocr} = id$$

$\equiv$ { .... }

$$\begin{cases} (id \times \pi_1) \cdot \mathsf{assocr} = \pi_1 \\ \pi_2 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \end{cases}$$

$\equiv$ { .... }

$$\begin{cases} \begin{cases} \pi_1 \cdot \mathsf{assocr} = \pi_1 \cdot \pi_1 \\ \pi_1 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \cdot \pi_1 \end{cases} \\ \pi_2 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \end{cases}$$

$\equiv$ { .... }

$$\begin{cases} \pi_1 \cdot \mathsf{assocr} = \pi_1 \cdot \pi_1 \\ \begin{cases} \pi_1 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \cdot \pi_1 \\ \pi_2 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \end{cases} \end{cases}$$

$\equiv$ { .... }

$$\begin{cases} \pi_1 \cdot \mathsf{assocr} = \pi_1 \cdot \pi_1 \\ \pi_2 \cdot \mathsf{assocr} = \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \end{cases}$$

$\equiv$ { .... }

$$\mathsf{assocr} = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle \qquad \text{(F1)}$$

2. (a) Codifique (F1) directamente em Haskell e verifique o comportamemto dessa função no GHCi; (b) De seguida, converta — por igualdade extensional — (F1) para notação Haskell *pointwise* que não recorra a nenhum combinador nem projecção e verifique no GHCi que as duas versões dão os mesmos resultados.

(a) Encode (F1) directly in Haskell and check its behavior in GHCi; (b) Then convert — by extensional equality — (F1) to pointwise *Haskell code dispensing with combinators or projections, and let GHCi check that both versions give the same results.*

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3. Recorde a propriedade universal do combinador $[f\,,g]$,

Recall the universal property of the $[f\,,g]$ combinator,

$$k = [f\,,g] \quad \equiv \quad \begin{cases} k \cdot i_1 = f \\ k \cdot i_2 = g \end{cases}$$

Demonstre a igualdade

Prove the equality

$$[\underline{k}\,,\underline{k}] = \underline{k} \tag{F2}$$

recorrendo à propriedade universal acima e a uma lei que qualquer função constante $\underline{k}$ satisfaz. (Ver no formulário.)

using the universal property given above and a law that any constant function $\underline{k}$ satisfies. (Check the reference sheet.)

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4. Os isomorfismos

Both isomorphisms of

$$A \times (B + C) \underset{\text{undistr}}{\overset{\text{distr}}{\cong}} A \times B + A \times C$$

estudados na aula teórica estão codificados na biblioteca $Cp.hs$. Supondo $A = String$, $B = \mathbb{B}$ e $C = \mathbb{Z}$, (a) aplique no GHCi undistr, alternativamente, aos pares $(\texttt{"CP"}, \textsc{True})$ ou $(\texttt{"LEI"}, 1)$; (b) verifique que $(\mathsf{distr} \cdot \mathsf{undistr})\ x = x$ para essas (e quaisquer outras) situações que possa testar.

studied in the theory class are encoded in the library $Cp.hs$. Assuming that $A = String$, $B = \mathbb{B}$ and $C = \mathbb{Z}$, (a) apply in GHCi undistr*, alternatively, to the pairs $(\texttt{"CP"}, \textsc{True})$ or $(\texttt{"LAW"}, 1)$; (b) check that $(\mathsf{distr} \cdot \mathsf{undistr})\ x = x$ for these (and any other) situations you can test.*

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5. Recorde a função

Recall

$$\alpha = [\langle \underline{\textsc{False}}, id \rangle\,, \langle \underline{\textsc{True}}, id \rangle]$$

da ficha anterior. Mostre, usando a propriedade *universal-+*, que α se pode escrever em Haskell da forma seguinte:

from the previous exercise sheet. Show, using the $+$-universal law*, that α can be written in pointwise Haskell as follows:*

$$\begin{aligned} \alpha\ (i_1\ a) &= (\textsc{False}, a) \\ \alpha\ (i_2\ a) &= (\textsc{True}, a) \end{aligned}$$

Codifique α e teste-a no GHCi, onde i_1 (resp. i_2) se escreve `Left` (resp. `Right`).

Encode α and test it on GHCi.

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6. Recorra às leis dos coprodutos para mostrar que a definição que conhece da função factorial,

Show by coproduct laws that the usual definition of the factorial function,

$$\begin{aligned} &fac\ 0 = 1 \\ &fac\ (n+1) = (n+1) * fac\ n \end{aligned}$$

é equivalente à equação seguinte

is equivalent the following equation,

$$fac \cdot [\underline{0}, \mathsf{succ}] = [\underline{1}, \mathsf{mul} \cdot \langle \mathsf{succ}, fac \rangle]$$

onde

where

$$\begin{aligned} \mathsf{succ}\ n &= n+1 \\ \mathsf{mul}\ (a, b) &= a * b \end{aligned}$$

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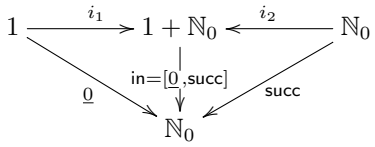
7. A função $\mathsf{in} = [\underline{0}, \mathsf{succ}]$ da questão anterior exprime, para $\mathsf{succ}\ n = n+1$, a forma como os números naturais ($\mathbb{N}_0$) são gerados a partir do número 0, de acordo com o diagrama seguinte:

Function $\mathsf{in} = [\underline{0}, \mathsf{succ}]$ *in the previous question (where* $\mathsf{succ}\ n = n+1$*) expresses the way natural numbers (*$\mathbb{N}_0$*) are generated from the number* 0*, according to the diagram below:*



(F3)

Sabendo que o tipo 1 coincide com o tipo $()$ em Haskell e é habitado por um único elemento, também designado por $()$, calcule a inversa de in,

Knowing that type 1 *matches type* $()$ *in Haskell and is inhabited by a single element, also denoted by* $()$*, find the inverse of* in*,*

$$\begin{cases} \mathsf{out}\ 0 = i_1\ () \\ \mathsf{out}\ (n+1) = i_2\ n \end{cases} \tag{F4}$$

resolvendo em ordem a out a equação

by solving the equation

$$\mathsf{out} \cdot \mathsf{in} = id \tag{F5}$$

e introduzindo variáveis.

for out *and adding variables.*

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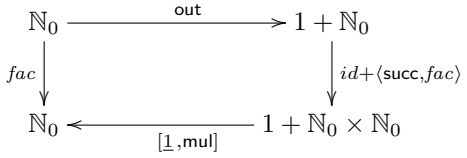
8. Verifique no GHCi que a seguinte função

On GHCi check that function

$$fac = [\underline{1}, \mathsf{mul}] \cdot (id + \langle \mathsf{succ}, fac \rangle) \cdot \mathsf{out}$$

a que corresponde o diagrama

captured by diagram



calcula o factorial da sua entrada, assumindo out (F4) e mul $(a, b) = a * b$ já definidas.

computes de factorial of its input, assuming out *(F4) and* mul $(a, b) = a * b$ *already defined.*

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9. **Questão prática** — Este problema não irá ser abordado em sala de aula. Os alunos devem tentar resolvê-lo em casa e, querendo, publicarem a sua solução no canal **#geral** do Slack, com vista à sua discussão com colegas.
   Os requisitos do problema são dados abaixo.
   NB: usa-se a notação X^* para designar o tipo $[X]$ em Haskell.

   Open assignment *— This assignment will not be addressed in class. Students should try to solve it at home and, wishing so, publish their solution in the* ***#geral*** *Slack channel, so that it can be discussed among colleagues.*
   The requirements of the problem are given below.
   NB*: notation X^* is used to denote the type $[X]$ in Haskell.*

   Problem requirements*:*

   The automatic generation of bibliographies in the LaTeX text preparation system is based bibliographic databases from which the following information can be extracted:

   $$Bib = (Key \times Aut^*)^*$$

   It associates authors (Aut) to citation keys (Key).
   Whenever LaTeX processes a text document, it compiles all occurrences of citation keys in an auxiliary file

   $$Aux = (Pag \times Key^*)^*$$

   associating pages (Pag) to the citation keys that occur in them.
   An ***author index*** *is an appendix to a text (e.g. book) indicating, in alphabetical order, the names of authors mentioned and the ordered list of pages where their works are cited, for example:*

   Arbib, M. A. – 10, 11
   Bird, R. – 28
   Horowitz, E. – 2, 3, 15, 16, 19
   Hudak, P. – 11, 12, 29
   Jones, C. B. – 3, 7, 28
   Manes, E. G. – 10, 11
   Sahni, S. – 2, 3, 15, 16, 19
   Spivey, J.M. – 3, 7
   Wadler, P. – 2, 3

   The above structure can be represented by the type

   $$Ind = (Aut \times Pag^*)^*$$

   listing authors (Aut) and the respective pages where they are mentioned (Pag).
   Write a Haskell function $mkInd : Bib \times Aux \to Ind$ that generates author indices (Ind) from Bib and Aux.
   Important*: Structure your solution across the $f \cdot g$, $\langle f, g \rangle$ and $f \times g$ combinators that can be found in library Cp.hs. Use* ***diagrams*** *to plan your proposed solution, which should avoid re-inventing functions over lists already available in the Haskell standard libraries.*
   □