Cálculo de Programas
Algebra of Programming

UNIVERSIDADE DO MINHO
Lic. em Engenharia Informática (3º ano)
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2022/23 - Ficha *(Exercise sheet)* nr. 6

1. Considere a seguinte sintaxe concreta em Haskell para um tipo que descreve pontos no espaço tridimensional:

   Consider the following concrete syntax in Haskell for a type that describes 3D-points:

$$\mathbf{data}\ Point\ a = Point\ \{x :: a, y :: a, z :: a\}\ \mathbf{deriving}\ (Eq, Show)$$

Pelo GHCi apura-se:

GHCi tells:

$$Point :: a \to a \to a \to Point\ a$$

Raciocinando apenas em termos de tipos, conjecture a definição de in na seguinte conversão dessa sintaxe concreta para abstracta:

Reasoning only in terms of types, conjecture the definition of in *in the following conversion from concrete to abstract syntax:*

$$\begin{tikzcd} Point\ A \arrow[r, bend left, "\mathsf{out} = \langle\langle x, y\rangle, z\rangle"] \arrow[r, phantom, "\cong"] & (A \times A) \times A \arrow[l, bend left, "\mathsf{in} = \cdots"] \end{tikzcd}$$

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2. Mostre que a propriedade de cancelamento da exponenciação

   Show that the cancellation property

$$\mathsf{ap} \cdot (\overline{f} \times id) = f \qquad \text{(F1)}$$

corresponde à definição

is nothing but the definition

$$\mathsf{curry}\ f\ a\ b = f\ (a, b)$$

quando se escreve curry f em lugar de $\overline{f}$.

once curry f *is written instead of* $\overline{f}$.

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3. Prove a igualdade

   Prove the equality

$$\overline{f \cdot (g \times h)} = \overline{\mathsf{ap} \cdot (id \times h)} \cdot \overline{f} \cdot g \qquad \text{(F2)}$$

usando as leis das exponenciais e dos produtos.

using the laws of products and exponentials.

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4. Mostre que

Show that

$$junc \cdot unjunc = id \tag{F3}$$
$$unjunc \cdot junc = id \tag{F4}$$

se verificam, onde

hold for

$$\begin{tikzcd} A^{B+C} \arrow[rr, bend left, "unjunc"] \arrow[rr, phantom, "\cong"] & & A^B \times A^C \arrow[ll, bend left, "junc"] \end{tikzcd} \qquad \left\{ \begin{array}{l} junc\ (f, g) = [f\ , g] \\ unjunc\ k = (k \cdot i_1, k \cdot i_2) \end{array} \right. \tag{F5}$$

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5. É dada a definição

Let flip *be defined by*

$$\mathsf{flip}\ f = \overline{\widehat{f} \cdot \mathsf{swap}} \tag{F6}$$

de acordo com:

according to:

$$\begin{array}{ccccccc} (C^B)^A & \cong & C^{A \times B} & \cong & C^{B \times A} & \cong & (C^A)^B \\ f & \mapsto & \widehat{f} & \mapsto & \widehat{f}.\mathsf{swap} & \mapsto & \overline{\widehat{f} \cdot \mathsf{swap}} = \mathsf{flip}\ f \end{array}$$

Mostre que flip é um isomorfismo por ser a sua própria inversa:

Show that it is an isomorphism because it is its own inverse:

$$\mathsf{flip}\ (\mathsf{flip}\ f) = f \tag{F7}$$

Mostre ainda que:

Furthermore show:

$$\mathsf{flip}\ f\ x\ y = f\ y\ x.$$

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6. O código que se segue, escrito em Haskell, implementa a noção de ciclo-for , onde b é o corpo ("body") do ciclo e i é a sua inicialização:

The following code in Haskell implements a for*-loop where b is the loop-body and i is its initialization:*

$$\left\{ \begin{array}{l} \mathsf{for}\ b\ i\ 0 = i \\ \mathsf{for}\ b\ i\ (n+1) = b\ (\mathsf{for}\ b\ i\ n) \end{array} \right. \tag{F8}$$

Mostre que $\mathsf{for}\ b\ i = (\!|\, g \,|\!)$ para um dado g (descubra qual) de acordo com o diagrama seguinte:

Show that $\mathsf{for}\ b\ i = (\!|\, g \,|\!)$ for some g (find which) according to the following diagram:

$$\begin{tikzcd} \mathbb{N}_0 \arrow[rr, bend left, "\mathsf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "\mathsf{for}\ b\ i"'] & & 1 + \mathbb{N}_0 \arrow[ll, bend left, "\mathsf{in}"] \arrow[d, "id + (\mathsf{for}\ b\ i)"] \\ B & & 1 + B \arrow[ll, bend left, "g"] \end{tikzcd} \qquad \text{where} \quad \left\{ \begin{array}{l} \mathsf{in} = [\mathsf{zero}\ , \mathsf{succ}] \\ \mathsf{zero}\ \_ = 0 \\ \mathsf{succ}\ n = n + 1 \end{array} \right.$$

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7. Mostre que $(a+)$ dada a seguir é o ciclo for $b\ i$ para um dado b e um dado i — descubra quais:

Show that $(a+)$ given next is a for-loop for $b\ i$ *for b and i to be calculated:*

$$\begin{cases} a + 0 = a \\ a + (n+1) = 1 + (a+n) \end{cases} \tag{F9}$$

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8. Apresente justificações para a seguinte dedução da lei de fusão-cata válida para ciclos-for:

Justify the following calculation of the cata-fusiona law valid for for-loops:

$$\begin{array}{cl}
& f \cdot (\!|\, g \,|\!) = (\!|\, h \,|\!) \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot (\!|\, g \,|\!) \cdot \mathsf{in} = h \cdot (id + f \cdot (\!|\, g \,|\!)) \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot g \cdot (id + (\!|\, g \,|\!)) = h \cdot (id + f \cdot (\!|\, g \,|\!)) \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot g \cdot (id + (\!|\, g \,|\!)) = h \cdot (id \cdot id + f \cdot (\!|\, g \,|\!)) \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot g \cdot (id + (\!|\, g \,|\!)) = h \cdot (id + f) \cdot (id + (\!|\, g \,|\!)) \\
\Leftarrow & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot g = h \cdot (id + f)
\end{array}$$

Em suma:

Summing up:

$$f \cdot (\!|\, g \,|\!) = (\!|\, h \,|\!) \quad \Leftarrow \quad f \cdot g = h \cdot (id + f) \tag{F10}$$

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9. Sabendo que $\mathsf{for}\ f\ i = (\!|\, [\underline{i}\ , f] \,|\!)$ — que é o que deve ter apurado na questão 6 —, recorra à lei de fusão-cata para demonstrar a propriedade:

Knowing that $\mathsf{for}\ f\ i = (\!|\, [\underline{i}\ , f] \,|\!)$ *— which is what you might have found in question 6 —, use the law of cata-fusion to prove the property:*

$$f \cdot (\mathsf{for}\ f\ i) = \mathsf{for}\ f\ (f\ i) \tag{F11}$$

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10. Mostre que as funções

Show that functions

$$\begin{array}{l} f = \mathsf{for}\ id\ i \\ g = \mathsf{for}\ \underline{i}\ i \end{array}$$

são a mesma função. (Qual?)

are the same function. (Which?)