Cálculo
de
Programas

Aula T01

J.N. Oliveira

From a mobile phone manufacturer

> *(...) For each* **list of calls** *stored in the mobile phone (eg. numbers dialed, SMS messages, lost calls), the* **store** *operation should work in a way such that* **(a)** *the more recently a call is made the more accessible it is;* **(b)** *no number appears twice in a list;* **(c)** *only the last 10 entries in each list are stored.*

From a mobile phone manufacturer

```
store’ ::  Call -> [Call] -> [Call]
store’ c l = take 10 (store c l)

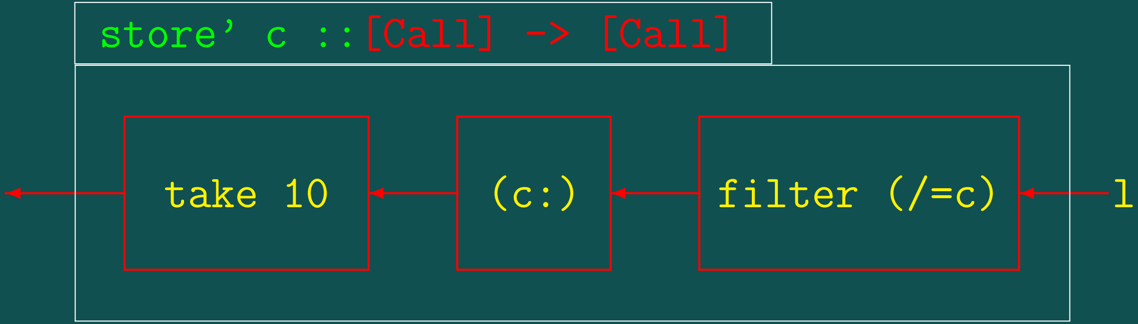
store ::  Call -> [Call] -> [Call]
store c l = c :  filter (/=c) l
```

From a mobile phone manufacturer

```
store’ ::  Call -> [Call] -> [Call]
store’ c l = take 10 (c :  filter (/=c) l)
```

From a mobile phone manufacturer

`store' c ::[Call] -> [Call]`



take 10

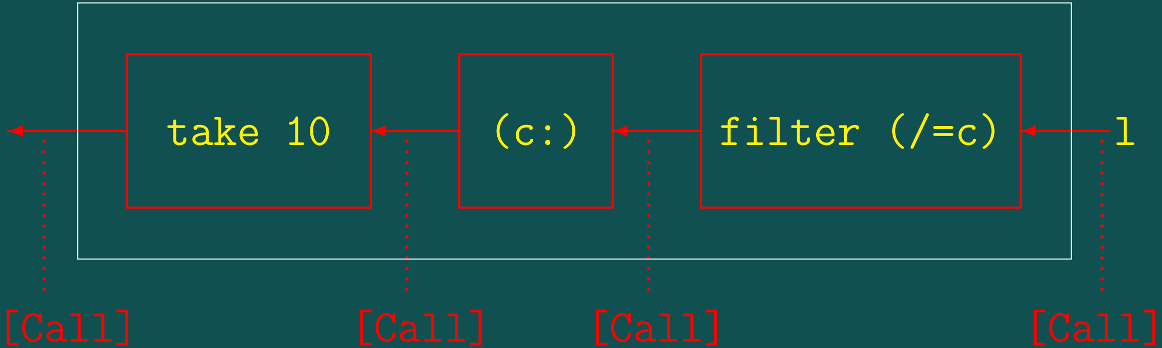
(c:)

filter (/=c)

l

From a mobile phone manufacturer

`store' c :: [Call] -> [Call]`



take 10

(c:)

filter (/=c)

l

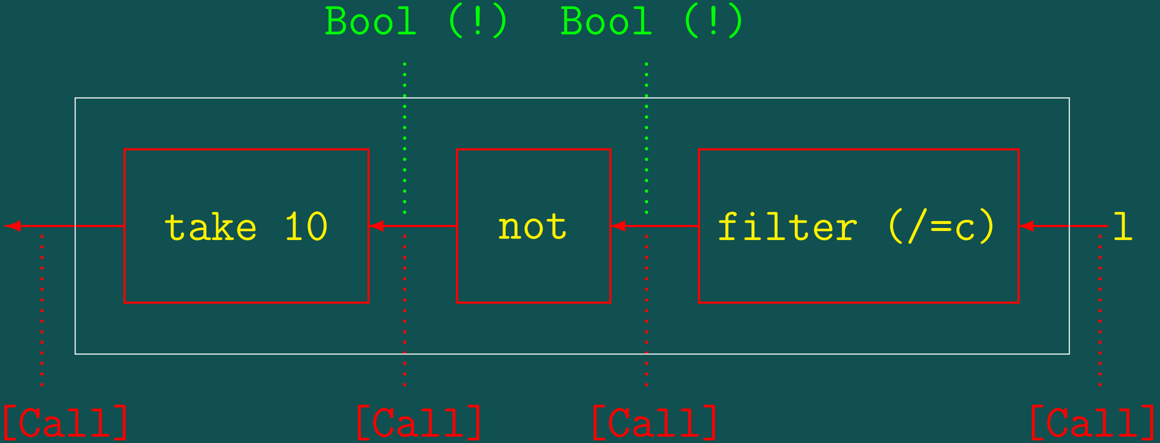
[Call]

[Call]

[Call]

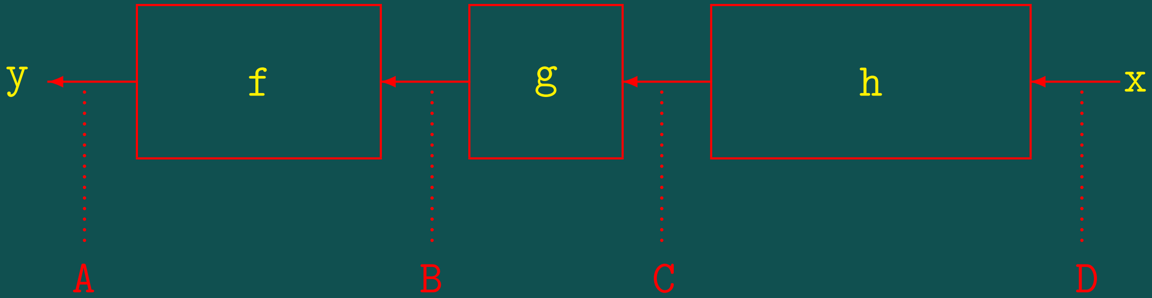
[Call]

Uups!



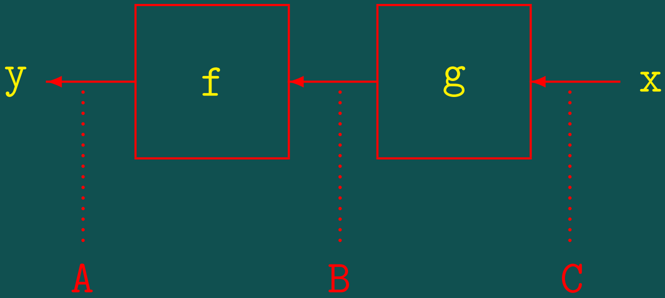
Em geral

`y = f(g(h x))`



Em geral

`y = f(g x)`



Simplificação

```
y = f(g x)
```

$$\begin{tikzcd}A & B \arrow[l, "f"'] & C \arrow[l, "g"']\end{tikzcd}$$

Composição

$$y = f(g\ x)$$

$$\begin{tikzcd}
A & B \arrow[l, "f"'] & C \arrow[l, "g"'] \\
A & & C \arrow[ll, "f \cdot g"']
\end{tikzcd}$$

$$y = (f \cdot g)\ x$$

Composição

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

$$f \cdot g \cdot h$$

$$a + b + c$$

Composição

$$store'\ c = take\ 10 \cdot \underbrace{(c:) \cdot filter\ (\neq c)}_{store\ c}$$

Composição

$$\mathit{store}'\ c = \mathit{take}\ 10 \cdot \underbrace{(c:) \cdot \mathit{filter}\ (\neq c)}_{\mathit{store}\ c}$$

isto é

$$\mathit{take}\ 10 \cdot ((c:) \cdot \mathit{filter}\ (\neq c))$$

Composição

$$\mathit{store}'\ c = \mathit{take}\ 10 \cdot \underbrace{(c:) \cdot \mathit{filter}\ (\neq c)}_{\mathit{store}\ c}$$

isto é

$$\mathit{take}\ 10 \cdot ((c:) \cdot \mathit{filter}\ (\neq c))$$

igual a

$$(\mathit{take}\ 10 \cdot (c:)) \cdot \mathit{filter}\ (\neq c)$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$
$$(a + b) + c \;=\; a + (b + c)$$

$$a + 0 = 0 + a = a$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

$$a + 0 = 0 + a = a$$

$$f \cdot ? = ? \cdot f = f$$

$$\begin{tikzcd}
A & B \arrow[l, "f"'] & C \arrow[l, "g"']
\end{tikzcd}$$

$$\begin{tikzcd}C \arrow[r, "g"] & B \arrow[r, "f"] & A\end{tikzcd}$$

$$\begin{tikzcd}
A & & \\
& B \arrow[ul, "f"'] & \\
& & C \arrow[ul, "g"']
\end{tikzcd}$$

A


A

$$\begin{tikzcd} A & A \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} A & A \arrow[l, "id"'] \end{tikzcd}$$

$$\begin{tikzcd}A & A \arrow[l, "id"']\end{tikzcd}$$

$$id\ a = a$$

Identidade

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \\
B
\end{tikzcd}$$

Identidade

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \\
B & B
\end{tikzcd}$$

Identidade

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \\
B & B \arrow[l, "id"]
\end{tikzcd}$$

Identidade

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[dl, "f"'] \arrow[d, "f"] \\
B & B \arrow[l, "id"]
\end{tikzcd}$$

Identidade

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[dl, "f"] \arrow[d, "f"] \\
B & B \arrow[l, "id"]
\end{tikzcd}$$

$$f \cdot id = f = id \cdot f$$

Composição e identidade

Associatividade:

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Composição e identidade

Associatividade:

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

“ Natural-*id*” :

$$f \cdot id = f = id \cdot f$$

$$f \cdot g$$

$$f \cdot g$$

$$f \times g\ ?$$

$$f \cdot g$$

$$f \times g \ ?$$

$$f + g \ ?$$

$C \xrightarrow{f} B$ e $A \xrightarrow{g} C$

$C \xrightarrow{f} B$ e $A \xrightarrow{g} C$

Composição: $A \xrightarrow{f \cdot g} B$

$$C \xrightarrow{f} B \quad \text{e} \quad A \xrightarrow{g} C$$

Composição: $A \xrightarrow{f \cdot g} B$

$$B \xleftarrow{f} C \quad \text{e} \quad C \xleftarrow{g} A$$

$$\begin{tikzcd}
D \arrow[r, "f"] & B \\
A \arrow[r, "g"] & C
\end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r, "f"] & B \\ A \arrow[r, "g"] & C \end{tikzcd}$$

$D = A$?

$$\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$f\ a\ \ldots\ g\ a$

$$\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$(f\ a, g\ a)$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

$$f\ a \in B$$

Produto cartesiano

$$A \times B = \left\{ (a, b) \mid a \in A \wedge b \in B \right\}$$

$$f\ a \in B$$
$$g\ a \in C$$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

$$\frac{f\ a \in B \qquad g\ a \in C}{(f\ a, g\ a) \in B \times C}$$

“Split”

$$\begin{tikzcd}
B & B \times C & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\langle f, g \rangle \; a = (f \; a, g \; a)$$

Produto

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Produto

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

$$\pi_1 : A \times B \to A$$
$$\pi_1\,(a, b) = a$$

Produto

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

$$\pi_1 : A \times B \to A$$
$$\pi_1\,(a, b) = a$$

$$\pi_2 : A \times B \to B$$
$$\pi_2\,(a, b) = b$$

Produto

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Produto

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\pi_1 \cdot \langle f, g \rangle = f$$

Produto

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\pi_1 \cdot \langle f, g \rangle = f$

$\pi_2 \cdot \langle f, g \rangle = g$

Produto

$$\langle f, g \rangle$$

Produto

$$\langle f, g \rangle$$

f e g em paralelo

f “split” g

Produto

$$\langle f, g \rangle$$

f e g em paralelo

f "split" g

$\langle f, g \rangle\ a = (f\ a, g\ a)$

Produto

$$\begin{tikzcd}
D \\
B \arrow[u, "f"]
\end{tikzcd}$$

Produto

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & & C \arrow[u, "g"']
\end{tikzcd}$$

Produto

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & C \arrow[u, "g"']
\end{tikzcd}$$

Produto

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[ul] \arrow[r, "\pi_2"'] & C \arrow[u, "g"']
\end{tikzcd}$$

Produto

$$\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[lu] \arrow[ru] & C \arrow[u, "g"']
\end{tikzcd}$$

Produto

$$\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}$$

Produto

$$\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "f \times g" description] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}$$

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$$

Produto

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "f \times g" description] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}$$

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$$

Sumário

$f \cdot g$

Sumário

$f \cdot g$ Composição sequencial

Sumário

$f \cdot g$

$\langle f, g \rangle$

Composição sequencial

Sumário

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela

Sumário

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)

Sumário

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	

Sumário

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela

Sumário

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela (assíncrona)

Sumário

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela (assíncrona)

Programação composicional

Cálculo
de
Programas

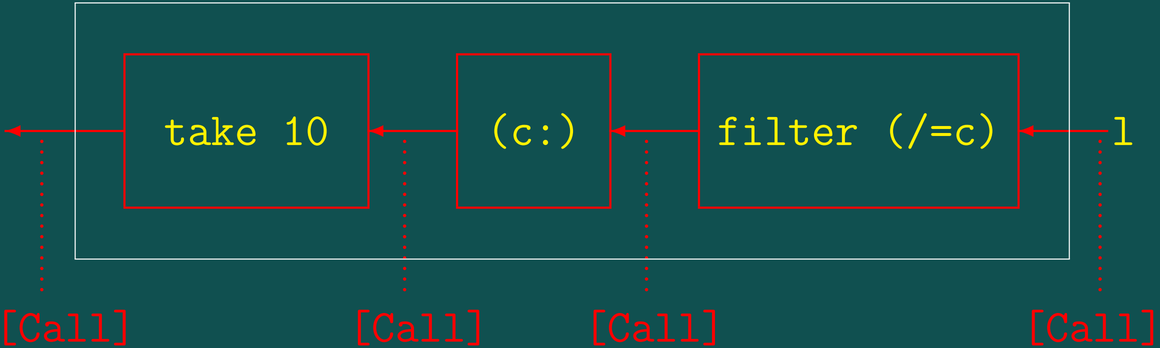
Aula T02

From a mobile phone manufacturer

(...) For each **list of calls** *stored in the mobile phone (eg. numbers dialed, SMS messages, lost calls), the* **store** *operation should work in a way such that* **(a)** *the more recently a call is made the more accessible it is;* **(b)** *no number appears twice in a list;* **(c)** *only the last 10 entries in each list are stored.*

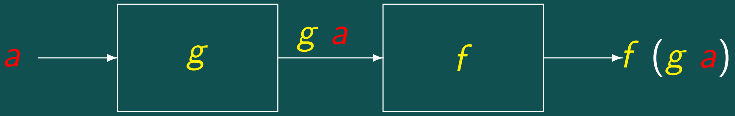
From a mobile phone manufacturer

`store' c :: [Call] -> [Call]`



Recapitulando

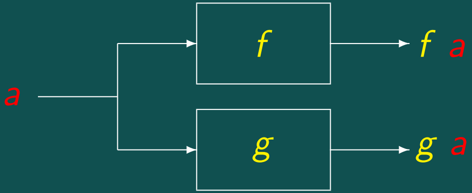
$$(f \cdot g)\ a = f\ (g\ a) \tag{2.6}$$



Composição de funções

Recapitulando

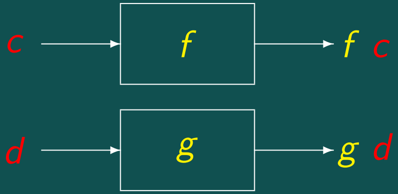
$$\langle f, g \rangle\ a = (f\ a, g\ a) \tag{2.20}$$



"Splits" de funções

Recapitulando

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle \tag{2.24}$$



c f $f\ c$

d g $g\ d$

Produtos de funções

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela (assíncrona)

Programação composicional

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle" description] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] \arrow[uul, "f \cdot h"] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle"] \arrow[ur, "g"'] & \\
 & D \arrow[u, "h"] \arrow[uul, "f \cdot h"] \arrow[uur, "g \cdot h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "f \cdot h"] \arrow[u, "\langle f , g \rangle \cdot h" description] \arrow[ur, "g \cdot h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "f \cdot h"] \arrow[u, "\langle f \cdot h, g \cdot h \rangle" description] \arrow[ur, "g \cdot h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[u, "\langle f , g \rangle \cdot h = \langle f \cdot h, g \cdot h \rangle" description] &
\end{tikzcd}$$

Fusão-$\times$

$$\langle f, g\rangle \cdot h = \langle f \cdot h, g \cdot h\rangle \tag{2.26}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g\rangle"] \arrow[ur, "g"'] & \\
 & D \arrow[uul, "f \cdot h"] \arrow[u, "h"] \arrow[uur, "g \cdot h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & D \times E & \\
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \\
 & A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[u, "\langle f , g \rangle" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E & \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"'] \\
 & A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[u, "\langle f , g \rangle"] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \arrow[u, "k"'] \\
 & A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[u, "\langle f, g \rangle" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[uul, "h \cdot f"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uul, "h \cdot f"'] \arrow[uur, "k \cdot g"] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "\langle h \cdot f, k \cdot g \rangle" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}$$

Absorção-$\times$

$$(h \times k) \cdot \langle f, g \rangle = \langle h \cdot f, k \cdot g \rangle \tag{2.27}$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
D \arrow[u, "h"] & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & E \arrow[u, "k"] \\
& C \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
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\end{tikzcd}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k" description] & C \arrow[u, "k"']
\end{tikzcd}$$

$$\pi_1 \cdot (h \times k) = h \cdot \pi_1$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"']
\end{tikzcd}$$

$$\pi_1 \cdot (h \times k) = h \cdot \pi_1 \qquad \pi_2 \cdot (h \times k) = k \cdot \pi_2$$

Natural-π_1, natural-π_2

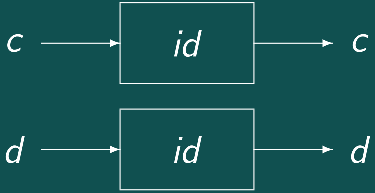
$$\pi_1 \cdot (h \times k) \ = \ h \cdot \pi_1 \tag{2.28}$$

$$\pi_2 \cdot (h \times k) = k \cdot \pi_2 \tag{2.29}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"'] & C \arrow[u, "k"']
\end{tikzcd}$$

Functor-id-$\times$

$$id \times id \;=\; id \tag{2.31}$$



Produto de identidades é a identidade.

Functor-$\times$

$$(f \times h) \cdot (g \times k) \;=\; (f \cdot g) \times (h \cdot k) \tag{2.30}$$

Composição de produtos é o **produto** das composições.

Duas leis que faltam

Reflexão-$\times$

$$\langle \pi_1, \pi_2 \rangle = \mathit{id} \tag{2.32}$$

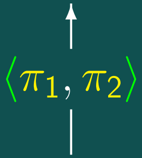
Duas leis que faltam

Reflexão-×

$$\langle \pi_1, \pi_2 \rangle = id \tag{2.32}$$

Eq-×

$$\langle i, j \rangle = \langle f, g \rangle \;\Leftrightarrow\; \left\{ \begin{array}{l} i = f \\ j = g \end{array} \right. \tag{2.64}$$



$\langle \pi_1, \pi_2 \rangle$

$$\begin{tikzcd} A & {} \\ & A \times B \arrow[ul, "\pi_1"] \arrow[u, "\langle \pi_1, \pi_2 \rangle" description] \end{tikzcd}$$

$$\begin{tikzcd}
A & {} & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[u, "\langle \pi_1, \pi_2 \rangle" description] &
\end{tikzcd}$$

$$\begin{tikzcd}
A & A \times B & B \\
& A \times B \arrow[ul, "\pi_1"'] \arrow[u, "\langle \pi_1 , \pi_2 \rangle" description] \arrow[ur, "\pi_2"] &
\end{tikzcd}$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, "\langle \pi_1, \pi_2 \rangle" description] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, "id" description] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

E agora o mais importante...

Recordar o **cancelamento**-$\times$:

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f,g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\pi_1 \cdot \langle f, g \rangle = f \qquad \pi_2 \cdot \langle f, g \rangle = g$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Rightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$\times$

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$\times$

Existência

$$k = \langle f, g \rangle \;\Rightarrow\; \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

"Existe uma solução — $k = \langle f, g \rangle$ — para as equações da direita"

Universal-$\times$

Unicidade

$$k = \langle f, g \rangle \quad \Leftarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

“As equações da direita só têm uma solução: $k = \langle f, g \rangle$”

Equações!

$$\begin{cases} x = 2\,y \\ z = \frac{y}{3} \\ x + y + z = 10 \end{cases}$$

Equações!

$$\begin{cases} x = 2\,y \\ z = \frac{y}{3} \\ x + y + z = 10 \end{cases} \Leftrightarrow \begin{cases} x = 6 \\ z = 1 \\ y = 3 \end{cases}$$

Equações!

Problema

Resolver a equação

$$\langle f , g \rangle = id$$

em ordem a f *e a* g.

Equações!

Resolução

Em $$k = \langle f, g \rangle \Leftrightarrow \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

Equações!

Resolução

Em $k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ fazer $k = id$

Equações!

Resolução

Em $k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ fazer $k = id$

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot id = f \\ \pi_2 \cdot id = g \end{cases}$$

Equações!

Resolução

Em $k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ fazer $k = id$

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Equações!

$$id = \langle f, g \rangle \iff \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Equações!

$$id = \langle f, g \rangle \iff \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Substituindo:

$$id = \langle \pi_1, \pi_2 \rangle$$

Equações!

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Substituindo:

$$id = \langle \pi_1, \pi_2 \rangle$$

Problema

Resolver a equação

$$\langle h, k \rangle = \langle f, g \rangle$$

Problema

Resolver a equação

$$\langle h, k \rangle = \langle f, g \rangle$$

(1 equação, 4 incógnitas)

Resolução

$$\langle h, k \rangle = \langle f, g \rangle$$

Resolução

$$
\begin{array}{ll}
 & \langle h, k \rangle = \langle f, g \rangle \\
\Leftrightarrow & \qquad \{ \ \text{universal-}\times \ \} \\
 & \left\{ \begin{array}{l} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{array} \right.
\end{array}
$$

Resolução

$$
\begin{aligned}
& \langle h, k \rangle = \langle f, g \rangle \\
\Leftrightarrow \quad & \quad \{ \text{ universal-}\times \} \\
& \begin{cases} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{cases} \\
\Leftrightarrow \quad & \quad \{ \text{ cancelamento-}\times \} \\
& \begin{cases} h = f \\ k = g \end{cases}
\end{aligned}
$$

Resolução

$$\langle h, k \rangle = \langle f, g \rangle$$

$$\Leftrightarrow \qquad \{ \text{ universal-}\times \text{ } \}$$

$$\begin{cases} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{cases}$$

$$\Leftrightarrow \qquad \{ \text{ cancelamento-}\times \text{ } \}$$

$$\begin{cases} h = f \\ k = g \end{cases}$$

Eq-× !

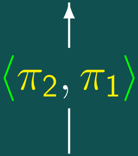
$$\langle \pi_1, \pi_2 \rangle = \mathit{id}$$

$\langle \pi_1, \pi_2 \rangle = \mathit{id}$

$\langle \pi_2, \pi_1 \rangle$?

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?



$\langle \pi_2, \pi_1 \rangle$

$$\langle \pi_1, \pi_2 \rangle = \mathit{id}$$

$$\langle \pi_2, \pi_1 \rangle ?$$

$$\begin{tikzcd}
B & {} \\
 & A \times B \arrow[ul, "\pi_2"] \arrow[u, "{\langle \pi_2, \pi_1 \rangle}"]
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & {} & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[ur, "\pi_1"'] \arrow[u, "\langle \pi_2, \pi_1 \rangle" description] &
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & B \times A & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & B \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle" description] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

Problema

Resolver

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

em ordem a k

Problema

Resolver

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

em ordem a k

$$\begin{tikzcd}
A \times B \arrow[r, bend left, "\langle \pi_2, \pi_1 \rangle"] & B \times A \arrow[l, bend left, "k"]
\end{tikzcd}$$

Resolução

$$\langle \pi_2, \pi_1 \rangle \cdot \mathit{k} = \mathit{id}$$

Resolução

$$
\begin{array}{ll}
 & \langle \pi_2, \pi_1 \rangle \cdot k = id \\
\Leftrightarrow & \qquad \{ \text{ fusão-}\times \ \} \\
 & \langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id
\end{array}
$$

Resolução

$$
\begin{array}{ll}
 & \langle \pi_2, \pi_1 \rangle \cdot k = id \\
\Leftrightarrow & \qquad \{ \text{ fusão-}\times \text{ } \} \\
 & \langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id \\
\Leftrightarrow & \qquad \{ \text{ universal-}\times \text{ } \} \\
 & \begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}
\end{array}
$$

Resolução

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

$$\Leftrightarrow \quad \{ \text{ fusão-}\times \}$$

$$\langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id$$

$$\Leftrightarrow \quad \{ \text{ universal-}\times \}$$

$$\begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}$$

$$\Leftrightarrow \quad \{ \text{ trivial } \}$$

$$\begin{cases} \pi_1 \cdot k = \pi_2 \\ \pi_2 \cdot k = \pi_1 \end{cases}$$

Resolução

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

$\Leftrightarrow$ { fusão-× }

$$\langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id$$

$\Leftrightarrow$ { universal-× }

$$\begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}$$

$\Leftrightarrow$ { trivial }

$$\begin{cases} \pi_1 \cdot k = \pi_2 \\ \pi_2 \cdot k = \pi_1 \end{cases}$$

$\Leftrightarrow$ { universal-× }

$$k = \langle \pi_2, \pi_1 \rangle$$

Swap

$$\begin{tikzcd}
A \times B \arrow[r, bend left, "\langle \pi_2, \pi_1 \rangle"] & B \times A \arrow[l, bend left, "\langle \pi_2, \pi_1 \rangle"]
\end{tikzcd}$$

$$swap = \langle \pi_2, \pi_1 \rangle$$

Swap

$$\begin{tikzcd}
A \times B \arrow[r, bend left, "\langle \pi_2, \pi_1 \rangle"] & B \times A \arrow[l, bend left, "\langle \pi_2, \pi_1 \rangle"]
\end{tikzcd}$$

$$swap = \langle \pi_2, \pi_1 \rangle$$

$$swap \cdot swap = id$$

Até agora

$f \cdot g$ Composição sequencial

Até agora

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Associatividade

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Associatividade

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associatividade?

$$\langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle ?$$

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Associatividade

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associatividade?

$$\langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle ?$$

Não! mas...

$\langle \langle f , g \rangle , h \rangle$

$$\begin{tikzcd} A & \\ & D \arrow[ul, "f"] \end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A & & B \\
& D \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\langle\langle f, g\rangle, h\rangle$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& D \arrow[ul, "f"] \arrow[u, "\langle f, g\rangle" description] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\langle \langle f, g \rangle, h \rangle$

$$\begin{tikzcd}
A \times B & & C \\
& D \arrow[ul, "\langle f, g \rangle"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & & C \\
& D \arrow[ul, "\langle f, g\rangle"] \arrow[u, "\langle\langle f, g\rangle, h\rangle" description] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle \langle f, g \rangle, h \rangle$

$$\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "{\langle f, g \rangle}"] \arrow[u, description, "{\langle \langle f, g \rangle, h \rangle}"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
B & & C \\
& D \arrow[ul, "g"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "g"] \arrow[u, "\langle g, h \rangle"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h\rangle\rangle$

$$\begin{tikzcd}
A & & B \times C \\
& D \arrow[ul, "f"] \arrow[ur, "\langle g, h\rangle"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
A & {} & B \times C \\
& D \arrow[ul, "f"] \arrow[u, "\langle f, \langle g, h \rangle \rangle" description] \arrow[ur, "\langle g, h \rangle"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
A & A \times (B \times C) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \times C \\
& D \arrow[ul, "f"] \arrow[u, "\langle f, \langle g, h \rangle \rangle" description] \arrow[ur, "\langle g, h \rangle"'] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "\langle f, g\rangle"] \arrow[u, "\langle\langle f, g\rangle, h\rangle"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "k"] & & A \times (B \times C) \\
& D \arrow[ul, "{\langle \langle f, g \rangle, h \rangle}"] \arrow[ur, "{\langle f, \langle g, h \rangle \rangle}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "k"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle\langle f,g\rangle,h\rangle"] \arrow[ur, "\langle f,\langle g,h\rangle\rangle"'] &
\end{tikzcd}$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \langle \langle f , g \rangle , h \rangle = \langle f , \langle g , h \rangle \rangle$$

$$k \cdot \underbrace{\langle \langle f , g \rangle , h \rangle}_{id?} = \langle f , \langle g , h \rangle \rangle$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \underbrace{\langle\langle f, g\rangle, h\rangle}_{id?} = \langle f, \langle g, h\rangle\rangle$$

Resolver $\langle\langle f, g\rangle, h\rangle = id$

$$\langle\langle f, g\rangle, h\rangle = id$$

$$\langle\langle f, g\rangle, h\rangle = id$$

$$\Leftrightarrow \quad \{\ \text{universal-}\times\ \}$$

$$\begin{cases} \pi_1 = \langle f, g\rangle \\ \pi_2 = h \end{cases}$$

$$
\begin{aligned}
& \langle \langle f , g \rangle , h \rangle = id \\
\Leftrightarrow \quad & \quad \{ \ \text{universal-}\times \ \} \\
& \begin{cases} \pi_1 = \langle f , g \rangle \\ \pi_2 = h \end{cases} \\
\Leftrightarrow \quad & \quad \{ \ \text{universal-}\times \ \} \\
& \begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}
\end{aligned}
$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k \cdot \underbrace{\langle\langle f, g\rangle, h\rangle}_{id} = \langle f, \langle g, h\rangle\rangle$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Melhorar...

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Melhorar...

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

$$\Leftrightarrow \quad \{ \ \pi_2 = id \cdot \pi_2 \ \}$$

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle$$

Melhorar...

$$
\begin{aligned}
& k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle \\
\Leftrightarrow \quad & \{ \ \pi_2 = id \cdot \pi_2 \ \} \\
& k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle \\
\Leftrightarrow \quad & \{ \ f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle \ \} \\
& k = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle
\end{aligned}
$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle \langle f, g \rangle, h \rangle"] \arrow[ur, "\langle f, \langle g, h \rangle \rangle"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle"] & & A \times (B \times C) \arrow[ll, bend left, "j"] \\
& D \arrow[ul, "\langle \langle f, g \rangle, h \rangle"] \arrow[ur, "\langle f, \langle g, h \rangle \rangle"'] &
\end{tikzcd}$$

$$\begin{tikzcd}(A \times B) \times C \arrow[r, bend left, "\langle \pi_1 \cdot \pi_1 , \pi_2 \times id \rangle"] & A \times (B \times C) \arrow[l, bend left, "\langle id \times \pi_1 , \pi_2 \cdot \pi_2 \rangle"]\end{tikzcd}$$

$$\begin{tikzcd}(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]\end{tikzcd}$$

$$assocr = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle$$
$$assocl = \langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle$$

$$\begin{tikzcd}(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]
\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Isomorfismo

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "assocr"] \arrow[r, phantom, "\cong"] & A \times (B \times C) \arrow[l, bend left, "assocl"]
\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Isomorfismo

$$\begin{tikzcd}
A \times B \arrow[rr, bend left, "\mathit{swap}"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "\mathit{swap}"]
\end{tikzcd}$$

$$\mathit{swap} \cdot \mathit{swap} = \mathit{id}$$

Isomorfismo

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, phantom, "\cong"] & B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$f \cdot g = id$$

$$g \cdot f = id$$

Isomorfismo

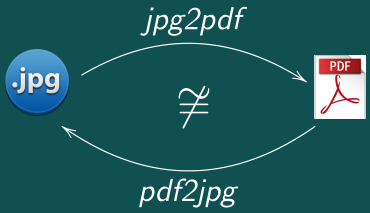
$$\underbrace{iso}_{a\ \ mesma}\ (\iota\sigma o)$$

Isomorfismo

$$\underbrace{\textit{iso}}_{\textit{a mesma}}\ (\iota\sigma o) + \underbrace{\textit{morfismo}}_{\textit{forma}}\ \ (\mu o\rho\phi\iota\sigma\mu o\zeta)$$

"Forma semelhante"

Problema prático!

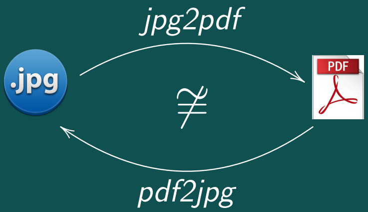


jpg2pdf

$\ncong$

pdf2jpg

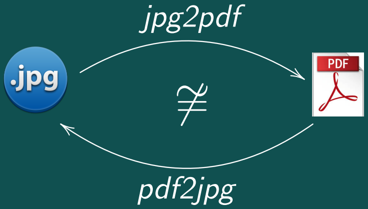
Problema prático!



$$\mathit{jpg2pdf} \cdot \mathit{pdf2jpg} \neq \mathit{id}$$

$$\mathit{pdf2jpg} \cdot \mathit{jpg2pdf} \neq \mathit{id}$$

Problema prático!



$$jpg2pdf \cdot pdf2jpg \neq id$$

$$pdf2jpg \cdot jpg2pdf \neq id$$

Conversão de formatos

Necessidade

$$\begin{tikzcd}A \arrow[d, "f"'] \\ B\end{tikzcd}$$

Conversão de formatos

Necessidade

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$$

Reutilizável

$$\begin{tikzcd} C \arrow[d, "r"] \\ D \end{tikzcd}$$

Conversão de formatos

Necessidade Reutilizável

$$\begin{tikzcd}
A \arrow[r, "exporta"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D
\end{tikzcd}$$

Conversão de formatos

$$\begin{tikzcd}
\text{Necessidade} & \text{Reutilizável} \\
A \arrow[r, "exporta"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D \arrow[l, "importa"]
\end{tikzcd}$$

Conversão de formatos

Necessidade Reutilizável

$$\begin{tikzcd}
A \arrow[r, "exporta"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D \arrow[l, "importa"]
\end{tikzcd}$$

$$f = importa \cdot r \cdot exporta$$

Conversão de formatos

$$\begin{tikzcd}
\text{Necessidade} & \text{Reutilizável} \\
A \arrow[r, "exporta"] \arrow[d, "f"'] \arrow[dr, phantom, "(\cong)"] & C \arrow[d, "r"] \\
B & D \arrow[l, "importa"]
\end{tikzcd}$$

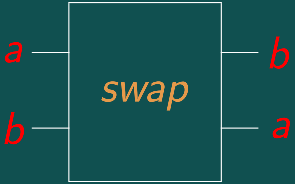
$$f = importa \cdot r \cdot exporta$$

A propósito de *swap*

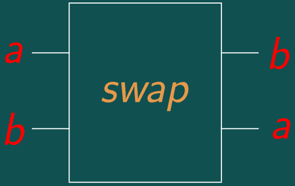
$$\begin{tikzcd}
A \times B \arrow[rr, bend left, "swap"] & \cong & B \times A \arrow[ll, bend left, "swap"]
\end{tikzcd}$$

Isomorfismos são computações **reversíveis**

A propósito de *swap*

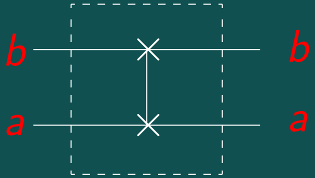


A propósito de *swap*



swap é uma das unidades básicas da **programação quântica**

A propósito de *swap*



Cálculo
de
Programas

Aula T03

Problema

Obter uma morada de um funcionário público, sabendo que este se pode identificar através do seu número do cartão de cidadão (CC) ou do seu número de identificação fiscal (NIF).

Problema

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$$morada : Iden \to Morada$$
$$Iden = CC \cup NIF$$

Problema!

$$CC = \mathbb{N}_0$$
$$NIF = \mathbb{N}_0$$

Problema!

$$CC = \mathbb{N}_0$$

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$$Iden = CC \cup NIF = \mathbb{N}_0 \cup \mathbb{N}_0 = \mathbb{N}_0$$

Problema!

$$CC = \mathbb{N}_0$$
$$NIF = \mathbb{N}_0$$
$$Iden = CC \cup NIF = \mathbb{N}_0 \cup \mathbb{N}_0 = \mathbb{N}_0$$

$$morada : \mathbb{N}_0 \to Morada \; (!)$$

Em geral, querendo

$$m : A \cup B \to C$$

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$$m : A \cup B \to C$$

Sabemos que

$$A \cup B = \{ a \mid a \in A \} \cup \{ b \mid b \in B \}$$

União disjunta

Precisamos de

$$\{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

União disjunta

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$$\{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

Definição

$$A + B = \{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

União disjunta

Tem-se:

$$A + B = \{\, i_1\ a \mid a \in A \,\} \cup \{\, i_2\ b \mid b \in B \,\}$$

definindo

$$i_1\ a = (1, a)$$
$$i_2\ b = (2, b)$$

União disjunta

Claramente:

$$m : A + B \to C$$

$$i_1 : A \to A + B$$

$$i_2 : B \to A + B$$

União disjunta

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A \arrow[r, "i_1"] & A + B \arrow[d, "m"] \\
 & C
\end{tikzcd}$$

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$$\begin{tikzcd}
A \arrow[r, "i_1"] & A + B \arrow[d, "m" description] & B \arrow[l, "i_2"'] \\
& C &
\end{tikzcd}$$

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A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "m"] & B \arrow[l, "i_2"'] \\
 & C &
\end{tikzcd}$$

$f = m \cdot i_1$

União disjunta

Claramente:

$$m : A + B \to C$$

$$i_1 : A \to A + B$$

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$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "m"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

$$f = m \cdot i_1 \qquad g = m \cdot i_2$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "{[f, g]}"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

Universal-$+$

$$k = [f, g] \Leftrightarrow \begin{cases} k \cdot i_1 = f \\ k \cdot i_2 = g \end{cases}$$

Universal-$+$

$$k = [f, g] \quad \Leftrightarrow \quad \left\{ \begin{array}{l} k \cdot i_1 = f \\ k \cdot i_2 = g \end{array} \right.$$

Comparar com

$$k = \langle f, g \rangle \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

Alternativa de funções

$$[f, g] : A + B \to C$$

$$[f, g]\; x \;=\; \begin{cases} x = i_1\; a \Rightarrow f\; a \\ x = i_2\; b \Rightarrow g\; b \end{cases}$$

Alternativa de funções

$$[f, g] : A + B \to C$$

$$[f, g]\ x \ = \ \begin{cases} x = i_1\ a \Rightarrow f\ a \\ x = i_2\ b \Rightarrow g\ b \end{cases}$$

$f + g$?

Soma de funções

$$\begin{tikzcd}
D \arrow[d, "f"'] \\
\mathsf{B}
\end{tikzcd}$$

Soma de funções

$$\begin{tikzcd} D \arrow[d, "f"'] & & E \arrow[d, "g"] \\ B & & C \end{tikzcd}$$

Soma de funções

$$\begin{tikzcd}
D \arrow[d, "f"'] & & E \arrow[d, "g"] \\
\mathrm{B} \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_2"]
\end{tikzcd}$$

Soma de funções

$$\begin{tikzcd}
D \arrow[d, "f"'] \arrow[dr] & & E \arrow[d, "g"] \\
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Soma de funções

$$\begin{tikzcd}
D \arrow[d, "f"'] \arrow[dr] & & E \arrow[dl] \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_2"]
\end{tikzcd}$$

Soma de funções

$$\begin{tikzcd}
D \arrow[r, "i_1"] \arrow[d, "f"'] \arrow[dr] & D + E \arrow[d, "f + g"] & E \arrow[l, "i_2"'] \arrow[d, "g"] \arrow[dl] \\
B \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_2"]
\end{tikzcd}$$

$$f + g = [i_1 \cdot f \, , \, i_2 \cdot g]$$

Coprodutos — leis

“É só virar as setas”, cf.

Absorção-$+$ $$[h, k] \cdot (f + g) = [h \cdot f , k \cdot g] \qquad (2.43)$$

Coprodutos — leis

"É só virar as setas", cf.

Absorção-$+$ $$[h, k] \cdot (f + g) = [h \cdot f, k \cdot g] \qquad (2.43)$$

Fusão-$+$ $$f \cdot [h, k] = [f \cdot h, f \cdot k] \qquad (2.42)$$

Coprodutos — leis

"É só virar as setas", cf.

Absorção-$+$ $$[h, k] \cdot (f + g) = [h \cdot f, k \cdot g] \qquad (2.43)$$

Fusão-$+$ $$f \cdot [h, k] = [f \cdot h, f \cdot k] \qquad (2.42)$$

Reflexão-$+$ $$[i_1, i_2] = id \qquad (2.41)$$

Coprodutos — leis

Igualdade-$+$
$$[h, k] = [f, g] \Leftrightarrow \left\{ \begin{array}{l} h = f \\ k = g \end{array} \right. \qquad (2.66)$$

Coprodutos — leis

Igualdade-$+$
$$[h, k] = [f, g] \Leftrightarrow \left\{ \begin{array}{l} h = f \\ k = g \end{array} \right. \tag{2.66}$$

Functor-$+$
$$(h + k) \cdot (f + g) = h \cdot f + k \cdot g \tag{2.44}$$

Coprodutos — leis

Igualdade-$+$
$$[h, k] = [f, g] \Leftrightarrow \left\{ \begin{array}{l} h = f \\ k = g \end{array} \right. \qquad (2.66)$$

Functor-$+$
$$(h + k) \cdot (f + g) = h \cdot f + k \cdot g \qquad (2.44)$$

Functor-id-$+$
$$id + id = id \qquad (2.45)$$

etc

Todas estas leis estão no formulário (Ver **Material** na página pública)

Sumário

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela
$f \times g$	Produto de funções

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Programação composicional

Será que...

$$f \times (g + h) \overset{?}{=} f \times g + f \times h$$

Será que...

$$f \times (g + h) \overset{?}{=} f \times g + f \times h$$

Ou que

$$A \times (B + C) \overset{?}{\cong} A \times B + A \times C$$

Será que...

$$f \times (g + h) \overset{?}{=} f \times g + f \times h$$

Ou que

$$A \times (B + C) \overset{?}{\cong} A \times B + A \times C$$

Haverá 0 tal que

Será que...

$$f \times (g + h) \overset{?}{=} f \times g + f \times h$$

Ou que

$$A \times (B + C) \overset{?}{\cong} A \times B + A \times C$$

Haverá 0 tal que

$$A \times 0 = 0 \quad ? \; A + 0 = A \quad ??$$

Relembrar

$$\begin{cases} id : A \to A \\ id\ a = a \end{cases}$$

$$\begin{tikzcd} A \arrow[r, bend left, "id"] \arrow[r, phantom, "\cong"] & A \arrow[l, bend left, "id"] \end{tikzcd}$$

Relembrar

$$\begin{cases} id : A \rightarrow A \\ id\ a = a \end{cases}$$

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Relembrar

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\end{tikzcd}$$

$$f \cdot g = id$$

$$g \cdot f = id$$

Relembrar

$$\begin{cases} swap : A \times B \rightarrow B \times A \\ swap\ (a, b) = (b, a) \end{cases}$$

Relembrar

$$\begin{cases} swap : A \times B \to B \times A \\ swap\,(a, b) = (b, a) \end{cases}$$

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "swap"] \arrow[r, phantom, "\cong"] & {} & B \times A \arrow[ll, bend left, "swap"] \end{tikzcd}$$

Relembrar

$$\begin{cases} swap : A \times B \to B \times A \\ swap\ (a, b) = (b, a) \end{cases}$$

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, bend left, "?"] \arrow[rr, phantom, "?"] & & A \times B \arrow[ll, bend left, "\pi_1"] \end{tikzcd}$$

Relembrar

$$\begin{cases} swap : A \times B \to B \times A \\ swap\ (a, b) = (b, a) \end{cases}$$

$$\begin{tikzcd}
A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, bend left, "?"] \arrow[rr, phantom, "?"] & & A \times B \arrow[ll, bend left, "\pi_1"]
\end{tikzcd}$$

$$\pi_1 \cdot \langle id, h \rangle = id$$

Relembrar

$$\left\{\begin{array}{l} swap : A \times B \to B \times A \\ swap\ (a, b) = (b, a) \end{array}\right.$$

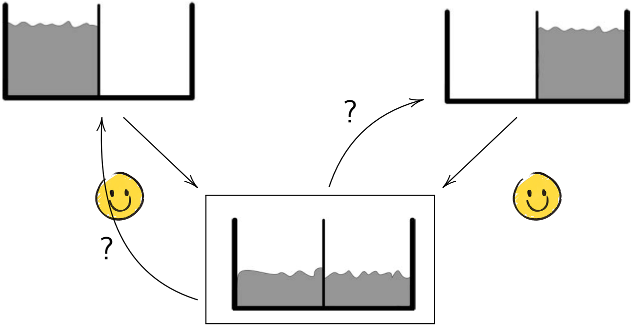
$$\begin{tikzcd}
A \times B \arrow[r, bend left, "swap"] \arrow[r, phantom, "\cong"] & B \times A \arrow[l, bend left, "swap"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, bend left, "?"] \arrow[r, phantom, "?"] & A \times B \arrow[l, bend left, "\pi_1"]
\end{tikzcd}$$

$$\pi_1 \cdot \langle id, h \rangle = id$$

$$\langle id, h \rangle \cdot \pi_1 \neq id$$

Irreversibilidade...



Rolf Landauer (1927–99)

“Information is physical”

Ainda pior que π_1...

$$\begin{cases} \mathit{zero}\ x = 0 \\ \mathit{one}\ x = 1 \end{cases}$$

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Ainda pior que π_1...

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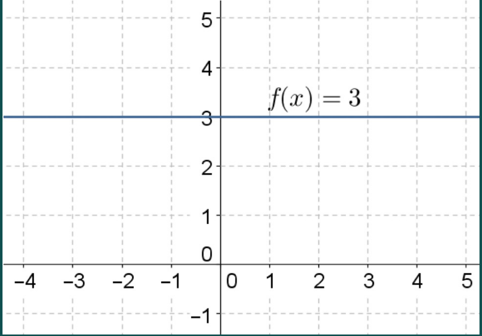
Ainda pior que π_1...

$$\begin{cases} \mathit{zero}\ x = 0 \\ \mathit{one}\ x = 1 \end{cases}$$

$$f\ x = 3$$



$$\mathit{one}\ 10 = 1$$

$$\mathit{one}\ \texttt{"string"} = 1$$

$$\mathit{zero}\ \texttt{"string"} = 0$$

Função constante

$3 \in \mathbb{N}_0$

$f\ x = 3$

Função constante

$3 \in \mathbb{N}_0$

$f\ x = 3$

$A \xrightarrow{f} \mathbb{N}_0$

Função constante

$3 \in \mathbb{N}_0$

$b_0 \in B$

$f\ x = 3$

$A \xrightarrow{f} \mathbb{N}_0$

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$$\begin{cases} \underline{b_0} : A \to B \\ \underline{b_0}\ a = b_0 \end{cases}$$

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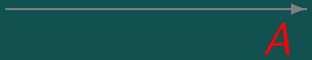

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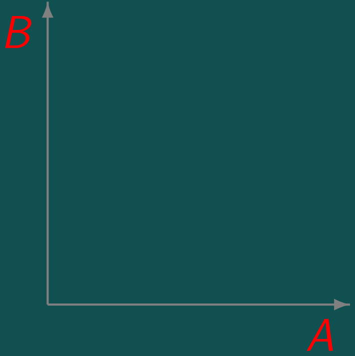

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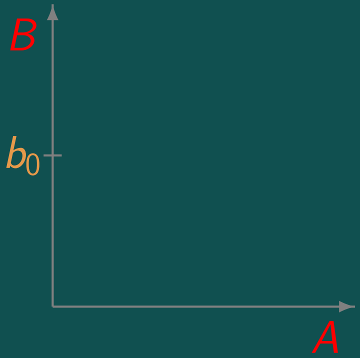

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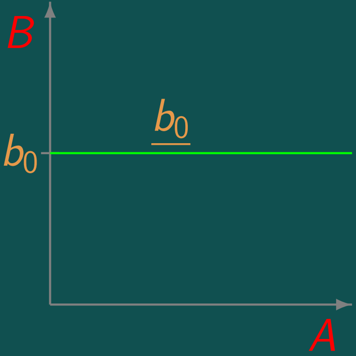

Função constante

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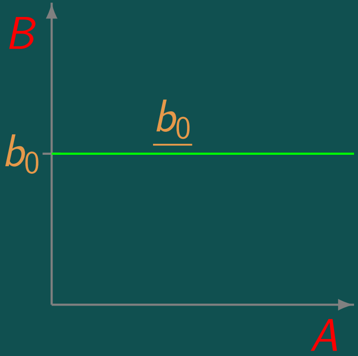

(**Haskell**: `const` b_0)

Propriedades

$$\underline{b} \cdot f = \underline{b}$$

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$$g \cdot \underline{b} = \underline{g \; b}$$

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$$\begin{tikzcd} B & A \arrow[l, "\underline{b}"'] \end{tikzcd}$$

Propriedades

$$\underline{b} \cdot f = \underline{b}$$

$$g \cdot \underline{b} = \underline{g\ b}$$

$$\begin{tikzcd}
 & C \arrow[d, "f"] \\
B & A \arrow[l, "\underline{b}"']
\end{tikzcd}$$

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B \arrow[d, "g"'] & A \arrow[l, "\underline{b}"'] \\
D &
\end{tikzcd}$$

Propriedades

$$\underline{b} \cdot f = \underline{b}$$

$$g \cdot \underline{b} = \underline{g\ b}$$

$$\begin{tikzcd}
 & C \arrow[dl, "\underline{b}"'] \arrow[d, "f"] \\
B \arrow[d, "g"'] & A \arrow[l, "\underline{b}"'] \arrow[dl, "\underline{g\ b}"] \\
D &
\end{tikzcd}$$

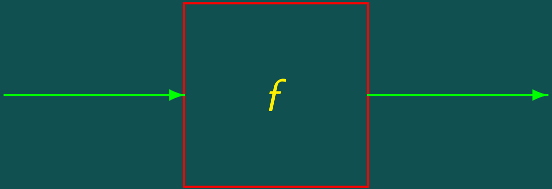
Propriedades

$$\underline{b} \cdot f = \underline{b}$$

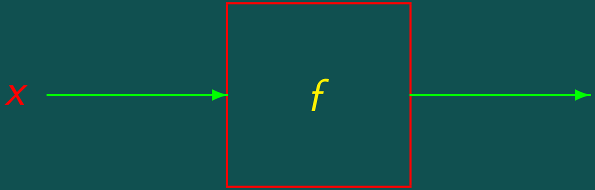
$$g \cdot \underline{b} = \underline{g \ \ b}$$

$$\begin{tikzcd}
& C \arrow[dl, "\underline{b}"'] \arrow[d, "f"] \\
B \arrow[d, "g"'] & A \arrow[l, "\underline{b}"'] \arrow[dl, "\underline{g \ \ b}"] \\
D &
\end{tikzcd}$$

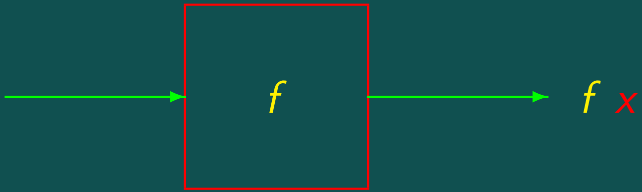
Gestão de informação



Gestão de informação

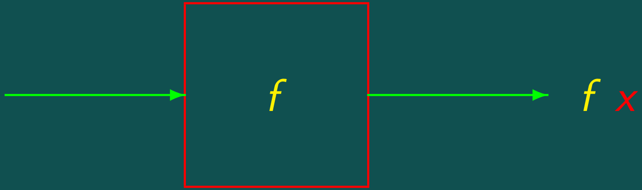


Gestão de informação

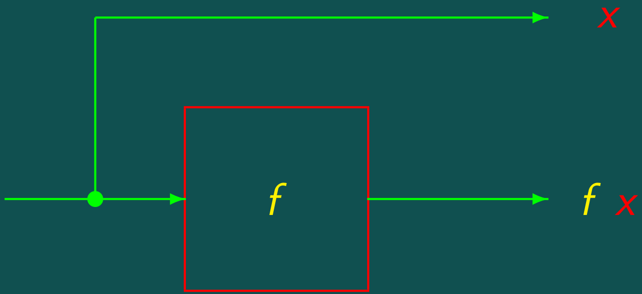


Gestão de informação

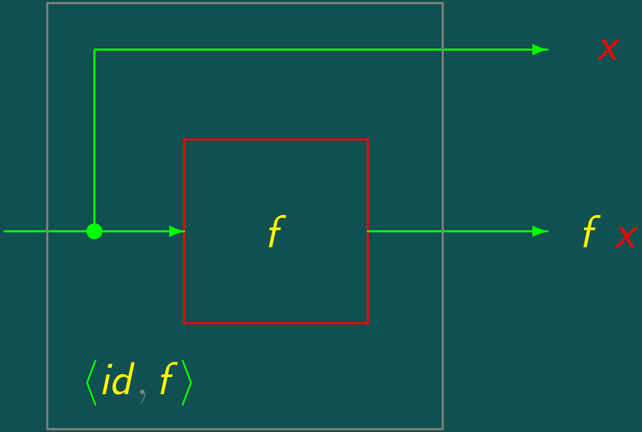
x?



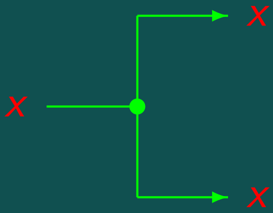
Gestão de informação



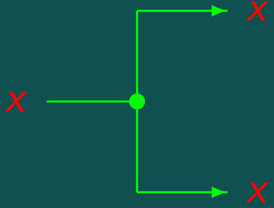
Gestão de informação



Duplicação de informação

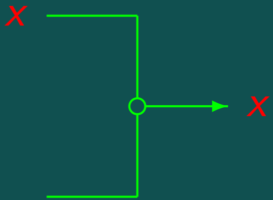


Duplicação de informação



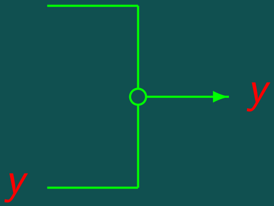
$$dup = \langle id, id \rangle$$

Junção de informação



$$join = [id\,,\,id]$$

Junção de informação

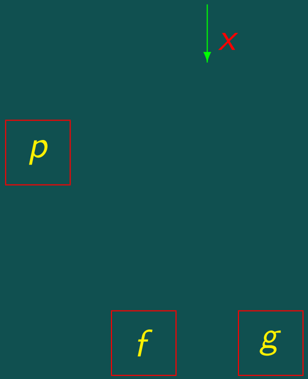


$$join = [id\,,\,id]$$

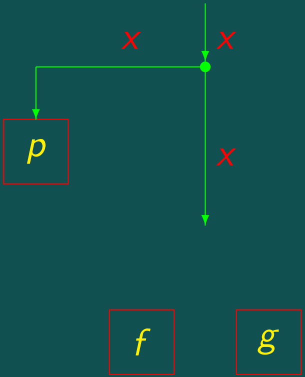
if-then-else

$$y = \textbf{if } p\ x \textbf{ then } f\ x \textbf{ else } g\ x$$

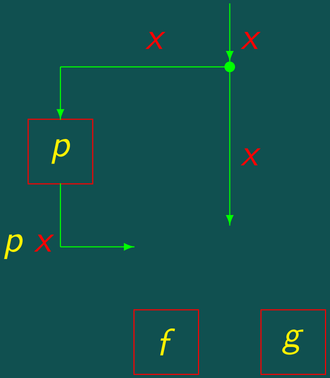
if-then-else



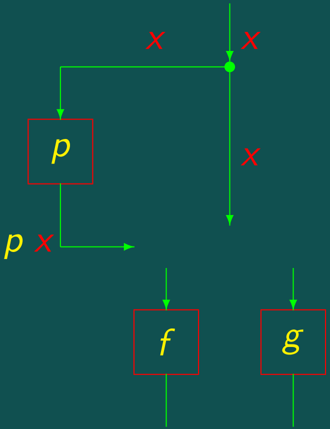
if-then-else



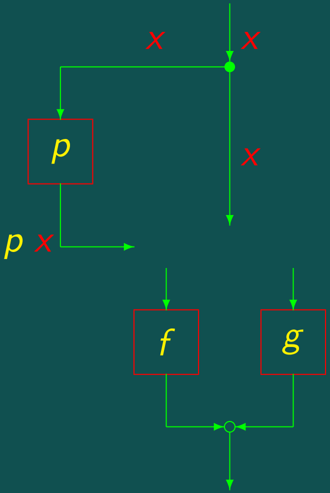
if-then-else



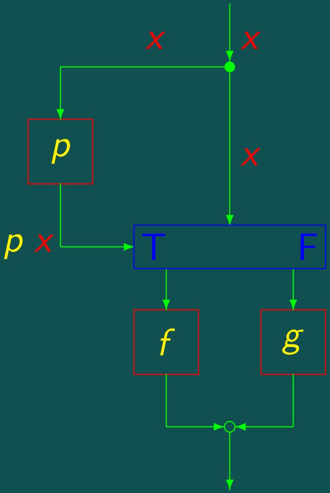
if-then-else



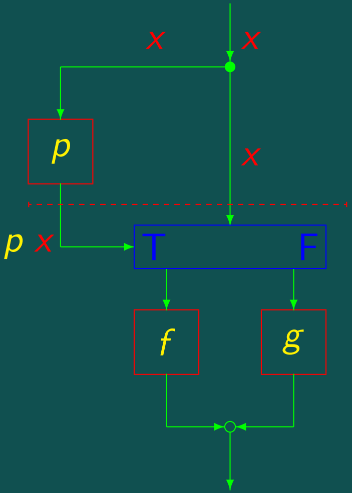
if-then-else



if-then-else



if-then-else



“Point-free”

if-then-else

$x \in A$

if-then-else

$$x \in A$$

$$(p\ x, x) \in \mathbb{B} \times A$$

if-then-else

$$x \in A$$

$$(p\ x, x) \in \mathbb{B} \times A$$

$$\mathbb{B} = \{F, T\}$$

if-then-else

$\mathbb{B}$ em Haskell:

```
data Bool = False | True
```

$x \in A$

$(p\ x, x) \in \mathbb{B} \times A$

$\mathbb{B} = \{F, T\}$

if-then-else

$x \in A$

$(p\ x, x) \in \mathbb{B} \times A$

$\mathbb{B} = \{F, T\}$

$\mathbb{B}$ em Haskell:

```
data Bool = False | True
```

T representado por `True`

F representado por `False`

O tipo 2

$$\mathbf{B} = \{\mathsf{F}, \mathsf{T}\} \cong \{\texttt{False}, \texttt{True}\}$$

O tipo 2

$$\mathbb{B} = \{F, T\} \cong \{\texttt{False}, \texttt{True}\} \cong \{0, 1\} \cong \cdots \cong 2$$

$$\begin{tikzcd}
\mathbb{B} \arrow[r, bend left, "g"] \arrow[r, phantom, "\cong"] & \{0, 1\} \arrow[l, bend left, "f"]
\end{tikzcd}$$

O tipo 2

$$\mathbb{B} = \{F, T\} \cong \{\texttt{False}, \texttt{True}\} \cong \{0, 1\} \cong \cdots \cong 2$$

$$\begin{tikzcd}
\mathbb{B} \arrow[r, bend left, "g"] \arrow[r, phantom, "\cong"] & \{0,1\} \arrow[l, bend left, "f"]
\end{tikzcd}$$

$$\begin{cases} f\ 0 = F \\ f\ 1 = T \end{cases}$$

O tipo 2

$$\mathbb{B} = \{F, T\} \cong \{\texttt{False}, \texttt{True}\} \cong \{0, 1\} \cong \cdots \cong 2$$

$$\begin{tikzcd}
\mathbb{B} \arrow[r, bend left, "g"] \arrow[r, phantom, "\cong"] & \{0, 1\} \arrow[l, bend left, "f"]
\end{tikzcd}$$

$$\begin{cases} f\ 0 = F \\ f\ 1 = T \end{cases}$$

$$\begin{cases} g\ F = 0 \\ g\ T = 1 \end{cases}$$

$$a + a = 2\,a$$

$$A + A \xrightarrow{join} A$$

$$a + a = 2\,a$$

$$A + A \xrightarrow{join} A$$

$$A \xrightarrow{\langle p, id \rangle} 2 \times A$$

$$a + a = 2\,a$$

$$A + A \xrightarrow{\mathit{join}} A$$

$$A \xrightarrow{\langle p, \mathit{id} \rangle} 2 \times A$$

Será que...

$$\begin{tikzcd} 2 \times A \arrow[r, bend left, "g"] \arrow[r, phantom, "\cong"] & A + A \arrow[l, bend left, "f"] \end{tikzcd}$$

$$\begin{tikzcd} 2 \times A \arrow[rr, bend left, "g"] \arrow[rr, phantom, "\cong"] && A + A \arrow[ll, bend left, "f"] \end{tikzcd}$$

$$f = [\langle \underline{T}, id \rangle, \langle \underline{F}, id \rangle]$$

$$g = \cdots$$

Notação

$$\begin{tikzcd} 2 \times A \arrow[r, bend left, "\alpha^\circ"] \arrow[r, phantom, "\cong"] & A + A \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

$$\alpha = [\langle \underline{T}, id \rangle, \langle \underline{F}, id \rangle]$$

$$\alpha^\circ = \cdots$$

Notação

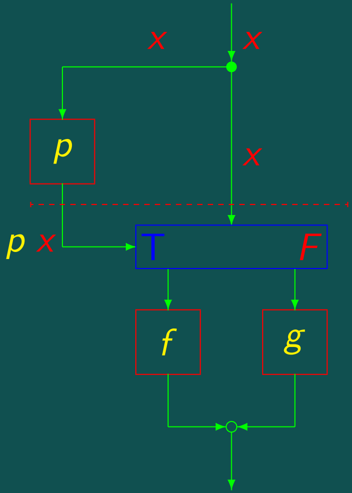
$$\begin{tikzcd} 2 \times A \arrow[rr, bend left, "\alpha^{\circ}"] \arrow[rr, phantom, "\cong"] & & A + A \arrow[ll, bend left, "\alpha"] \end{tikzcd}$$

$$\alpha \cdot \alpha^{\circ} = id$$

$$\alpha^{\circ} \cdot \alpha = id$$

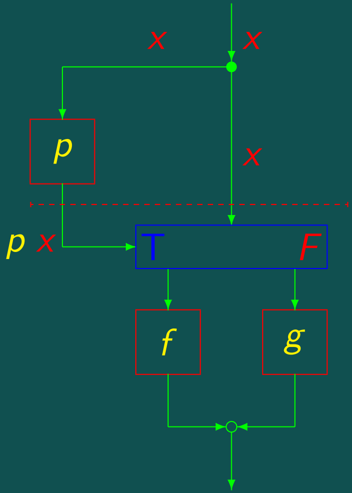
α determina α°

if-then-else



if-then-else

$$\begin{tikzcd}
A \arrow[d, "\langle p, id\rangle"] \\
2 \times A
\end{tikzcd}$$

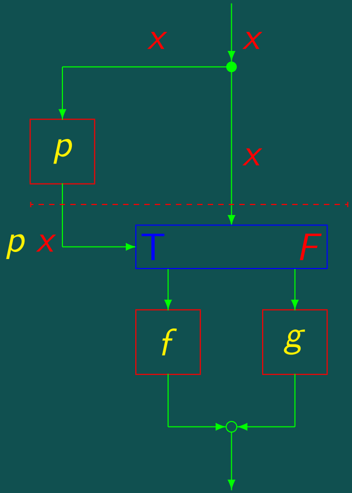


if-then-else

$$\begin{tikzcd} A \arrow[d, "\langle p, id \rangle"] \\ 2 \times A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$$

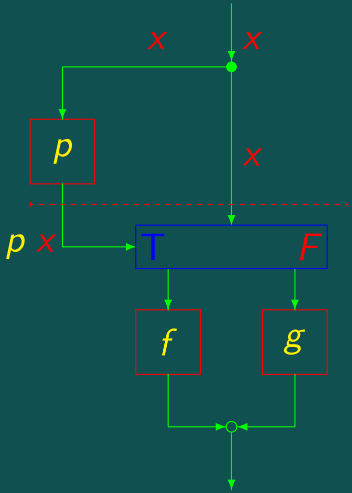
$$\begin{tikzcd} A \arrow[d, "g"] \\ B \end{tikzcd}$$



if-then-else

$$\begin{tikzcd}
A \arrow[d, "\langle p, id \rangle"] \\
2 \times A
\end{tikzcd}$$

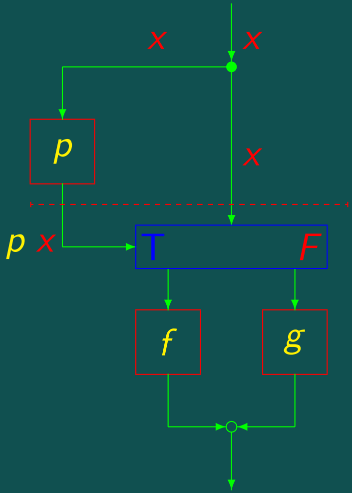
$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "f"'] & A + A \arrow[d, dashed, "f + g"] & A \arrow[l, "i_2"'] \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & B + B & B \arrow[l, "i_2"]
\end{tikzcd}$$



if-then-else

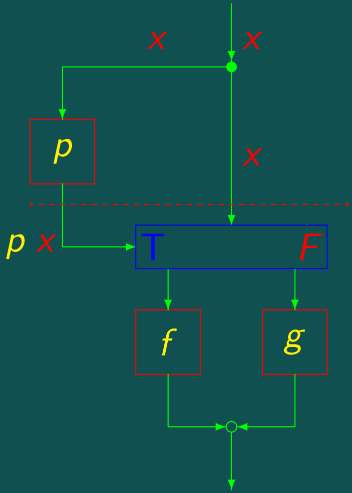
$$\begin{tikzcd}
A \arrow[d, "\langle p, id \rangle"] \\
2 \times A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "f"'] & A + A \arrow[d, dashed, "f + g"] & A \arrow[l, "i_2"'] \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & B + B \arrow[d, "join"] & B \arrow[l, "i_2"] \\
 & B &
\end{tikzcd}$$



if-then-else

$$\begin{tikzcd}
 & A \arrow[d, "\langle p, id \rangle"] & \\
 & 2 \times A \arrow[d, "\alpha^\circ"] & \\
A \arrow[r, "i_1"] \arrow[d, "f"'] & A + A \arrow[d, "f + g"] & A \arrow[l, "i_2"'] \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & B + B \arrow[d, "join"] & B \arrow[l, "i_2"] \\
 & B &
\end{tikzcd}$$



Condicional de McCarthy

$p \rightarrow f \; , \; g$ ”se p então f senão g”

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Do diagrama: $p \rightarrow f \; , \; g \quad = \quad join \cdot (f + g) \cdot \alpha^{\circ} \cdot \langle p, id \rangle$

Condicional de McCarthy

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Ora $join \cdot (f + g) = [id \, , id] \cdot (f + g) = [f \, , g]$

Condicional de McCarthy

$p \rightarrow f\ ,\ g$ ”se p então f senão g”

Do diagrama: $p \rightarrow f\ ,\ g \ = \ join \cdot (f + g) \cdot \alpha^{\circ} \cdot \langle p, id \rangle$

Ora $join \cdot (f + g) = [id\ ,\ id] \cdot (f + g) = [f\ ,\ g]$

Logo:

$$p \rightarrow f\ ,\ g \ = \ [f, g] \cdot \underbrace{\alpha^{\circ} \cdot \langle p, id \rangle}_{p?}$$

Condicional de McCarthy

Guarda de p:

$$\begin{tikzcd} A \arrow[r, "\langle p, id \rangle"] \arrow[rr, bend right, "p?"'] & 2 \times A \arrow[r, "\alpha^{\circ}"] & A + A \end{tikzcd}$$

Condicional:

$$p \rightarrow f\ ,\ g \ = \ [f\,, g] \cdot p?$$

Pointwise

$$p? : A \to A + A$$

$$p?\ a = \begin{cases} p\ a \Rightarrow i_1\ a \\ \neg\,(p\ a) \Rightarrow i_2\ a \end{cases}$$

Pointwise

$$p? : A \to A + A$$

$$p?\ a = \begin{cases} p\ a \Rightarrow i_1\ a \\ \neg\,(p\ a) \Rightarrow i_2\ a \end{cases}$$

daí

$$(p \to f\ ,\ g)\ a = \begin{cases} p\ a \Rightarrow f\ a \\ \neg\,(p\ a) \Rightarrow g\ a \end{cases}$$

Pointwise

$$p? : A \to A + A$$

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daí

$$(p \to f\ ,\ g)\ a = \begin{cases} p\ a \Rightarrow f\ a \\ \neg\,(p\ a) \Rightarrow g\ a \end{cases}$$

Análise por casos?

Pointwise

$$p? : A \to A + A$$

$$p?\ a = \begin{cases} p\ a \Rightarrow i_1\ a \\ \neg\ (p\ a) \Rightarrow i_2\ a \end{cases}$$

daí

$$(p \to f\ ,\ g)\ a = \begin{cases} p\ a \Rightarrow f\ a \\ \neg\ (p\ a) \Rightarrow g\ a \end{cases}$$

Análise por casos? Não!

“Química” entre condicional e composição?

$$h \cdot (p \rightarrow f\ ,\ g) = ?$$

“Química” entre condicional e composição?

$$h \cdot (p \rightarrow f \; , \; g) = ?$$

$$(p \rightarrow f \; , \; g) \cdot h = ?$$

“Calculemus”

$$h \cdot (p \rightarrow f\ ,\ g)$$

“Calculemus”

$$h \cdot (p \rightarrow f\ ,\ g)$$

$$\Leftrightarrow \qquad \{\ \text{definição do condicional}\ \}$$

$$h \cdot [f\,,g] \cdot p?$$

"Calculemus"

$$
\begin{aligned}
& h \cdot (p \rightarrow f\ ,\ g) \\
\Leftrightarrow \quad & \{\ \text{definição do condicional}\ \} \\
& h \cdot [f\,,g] \cdot p? \\
\Leftrightarrow \quad & \{\ \text{fusão-}+\ \} \\
& [h \cdot f\,, h \cdot g] \cdot p?
\end{aligned}
$$

“Calculemus”

$$
\begin{array}{ll}
 & h \cdot (p \rightarrow f \ , \ g) \\
\Leftrightarrow & \quad \{ \text{ definição do condicional } \} \\
 & h \cdot [f \, , g] \cdot p? \\
\Leftrightarrow & \quad \{ \text{ fusão-}+ \} \\
 & [h \cdot f \, , h \cdot g] \cdot p? \\
\Leftrightarrow & \quad \{ \text{ definição do condicional } \} \\
 & p \rightarrow (h \cdot f) \ , \ (h \cdot g)
\end{array}
$$

1ª lei de fusão do condicional

Logo:

$$h \cdot (p \rightarrow f\ ,\ g) \ = \ p \rightarrow (h \cdot f)\ ,\ (h \cdot g)$$

1ª lei de fusão do condicional

Logo:

$$h \cdot (p \rightarrow f \; , \; g) \; = \; p \rightarrow (h \cdot f) \; , \; (h \cdot g)$$

E se h correr **antes** do condicional?

$$(p \rightarrow f \; , \; g) \cdot h = ?$$

2ª lei de fusão do condicional

Ter-se-á:

$$(p \rightarrow f\ ,\ g) \cdot h \ = \ (p \cdot h) \rightarrow (f \cdot h)\ ,\ (g \cdot h)$$

2ª lei de fusão do condicional

Ter-se-á:

$$(p \rightarrow f \;,\; g) \cdot h \;=\; (p \cdot h) \rightarrow (f \cdot h) \;,\; (g \cdot h)$$

Intuitivamente 🙂

2ª lei de fusão do condicional

Ter-se-á:

$$(p \rightarrow f\ ,\ g) \cdot h \ = \ (p \cdot h) \rightarrow (f \cdot h)\ ,\ (g \cdot h)$$

Intuitivamente 🙂

Prova?

2ª lei de fusão do condicional

Ter-se-á:

$$(p \to f\ ,\ g) \cdot h \;=\; (p \cdot h) \to (f \cdot h)\ ,\ (g \cdot h)$$

Intuitivamente 🙂

Prova? Só mais tarde.

(precisamos de saber algo mais sobre p?)

John McCarthy (1927-2011)

Prémio Turing

Um dos *pais fundadores* da **Inteligência Artificial**

Doutorado em matemática (Princeton, 1951)

Inventor da linguagem **LISP**

Isomorfismos

$$A \times (B \times C) \cong (A \times B) \times C$$ 🙂

$$A \times B \cong B \times A$$ 🙂

Isomorfismos

$$A \times (B \times C) \cong (A \times B) \times C$$ 🙂

$$A \times B \cong B \times A$$ 🙂

$$A + (B + C) \cong (A + B) + C$$

$$A + B \cong B + A$$

Isomorfismos

$$A + A \cong 2 \times A$$ 🙂

Isomorfismos

$$A + A \cong 2 \times A$$ 🙂

$$A \times A \cong A^2$$?

Isomorfismos

$$A + A \cong 2 \times A$$ 🙂

$$A \times A \cong A^2$$?

$$A \times 1 \cong A$$?

Isomorfismos

$A + A \cong 2 \times A$ 🙂

$A \times A \cong A^2$?

$A \times 1 \cong A$?

$A \times 0 \cong 0$?

Isomorfismos

$$A + A \cong 2 \times A$$ 🙂

$$A \times A \cong A^2$$?

$$A \times 1 \cong A$$?

$$A \times 0 \cong 0$$?

$$A + 0 \cong A$$?

Isomorfismos

$$A + A \cong 2 \times A$$ 🙂

$$A \times A \cong A^2$$?

$$A \times 1 \cong A$$?

$$A \times 0 \cong 0$$?

$$A + 0 \cong A$$?

$$A \times (B + C) \cong A \times B + A \times C$$?

Distributividade

$$\begin{tikzcd}
A \times (B + C) \arrow[r, bend left, "\mathit{distr}"] \arrow[r, phantom, "\cong"] & A \times B + A \times C \arrow[l, bend left, "\mathit{undistr}"]
\end{tikzcd}$$

Distributividade

$$\begin{tikzcd}
A \times (B + C) \arrow[rr, bend left, "distr"] \arrow[rr, phantom, "\cong"] & & A \times B + A \times C \arrow[ll, bend left, "undistr"]
\end{tikzcd}$$

$$\begin{tikzcd}
(B + C) \times A \arrow[rr, bend left, "distl"] \arrow[rr, phantom, "\cong"] & & B \times A + C \times A \arrow[ll, bend left, "undistl"]
\end{tikzcd}$$

Lei da troca

Lei da troca

$$\begin{tikzcd}
A + B \arrow[d, "\alpha"] \\
{}
\end{tikzcd}$$

Lei da troca

$$\begin{tikzcd}
A + B \arrow[d, "\alpha"] \\
C \times D
\end{tikzcd}$$

Lei da troca

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "i"'] & A + B \arrow[d, "{[i , j]}" description] & B \arrow[l, "i_2"'] \arrow[dl, "j"] \\
 & C \times D &
\end{tikzcd}$$

Lei da troca

$$\begin{tikzcd}
 & A + B \arrow[dl, "m"'] \arrow[d, "\langle m, n \rangle"] \arrow[dr, "n"] & \\
C & C \times D \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & D
\end{tikzcd}$$

Lei da troca

$$\begin{tikzcd}
A \arrow[r, "i_1"] & A+B \arrow[dl, "{m=[f,g]}"'] \arrow[d, "{\langle m,n\rangle}" description] \arrow[dr, "{n=[h,k]}"] & B \arrow[l, "i_2"'] \\
C & C\times D \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & D
\end{tikzcd}$$

Lei da troca

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "i = \langle f, h \rangle"'] & A + B \arrow[d, "{[i, j]}" description] & B \arrow[l, "i_2"'] \arrow[dl, "j = \langle g, k \rangle"] \\
C & C \times D \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & D
\end{tikzcd}$$

Em suma

Lei da troca:

$$\langle [f, g], [h, k] \rangle = [\langle f, h \rangle, \langle g, k \rangle]$$

"tenho um split e dava-me jeito uma alternativa..."

...e vice-versa

Distributividade

$$\begin{tikzcd}
A \times (B + C) \arrow[rr, bend left, "\mathit{distr}"] & \cong & A \times B + A \times C \arrow[ll, bend left, "\mathit{undistr}"]
\end{tikzcd}$$

Distributividade

$$\begin{tikzcd}
A \times (B + C) \arrow[r, bend left, "distr"] \arrow[r, phantom, "\cong"] & A \times B + A \times C \arrow[l, bend left, "undistr"]
\end{tikzcd}$$

$$undistr = [\langle f, g \rangle, \langle h, k \rangle]$$

Distributividade

$$\begin{tikzcd}
A \times (B + C) \arrow[rr, bend left, "distr"] & \cong & A \times B + A \times C \arrow[ll, bend left, "undistr"]
\end{tikzcd}$$

$$undistr = [\langle f, g\rangle, \langle h, k\rangle]$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "f"']
\end{tikzcd}$$

Distributividade

$$\begin{tikzcd} A \times (B + C) \arrow[rr, bend left, "distr"] & \cong & A \times B + A \times C \arrow[ll, bend left, "undistr"] \end{tikzcd}$$

$$undistr = [\langle f, g\rangle, \langle h, k\rangle]$$

$$A \xleftarrow{f} A \times B$$

$$B + C \xleftarrow{g} A \times B$$

Distributividade

$$\begin{tikzcd}
A \times (B + C) \arrow[r, phantom, "\cong"] \arrow[r, bend left, "distr"] & A \times B + A \times C \arrow[l, bend left, "undistr"]
\end{tikzcd}$$

$$undistr = [\langle f, g\rangle, \langle h, k\rangle]$$

$$A \xleftarrow{f} A \times B \qquad\qquad A \xleftarrow{h} A \times C$$

$$B + C \xleftarrow{g} A \times B$$

Distributividade

$$\begin{tikzcd} A \times (B + C) \arrow[rr, bend left, "distr"] & \cong & A \times B + A \times C \arrow[ll, bend left, "undistr"] \end{tikzcd}$$

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Distributividade

$$undistr = [\langle f, g\rangle, \langle h, k\rangle]$$

$$\begin{tikzcd} A & A \times B \arrow[l, "f"'] \\ B + C & A \times B \arrow[l, "g"'] \end{tikzcd} \qquad \begin{tikzcd} A & A \times C \arrow[l, "h"'] \\ B + C & A \times C \arrow[l, "k"'] \end{tikzcd}$$

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$$undistr = [\langle f, g\rangle, \langle h, k\rangle]$$

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$$A \xleftarrow{\pi_1} A \times B$$

Distributividade

$$undistr = [\langle f, g\rangle, \langle h, k\rangle]$$

$$A \xleftarrow{f} A \times B$$

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$$A \xleftarrow{h} A \times C$$

$$B + C \xleftarrow{k} A \times C$$

$$A \xleftarrow{\pi_1} A \times B$$

$$B + C \xleftarrow{i_1 \cdot \pi_2} A \times B$$

Distributividade

$$undistr = [\langle f, g \rangle, \langle h, k \rangle]$$

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$$B + C \xleftarrow{i_1 \cdot \pi_2} A \times B \qquad B + C \xleftarrow{i_2 \cdot \pi_2} A \times C$$

$$undistr = [\langle \pi_1, i_1 \cdot \pi_2 \rangle, \langle \pi_1, i_2 \cdot \pi_2 \rangle]$$

Distributividade

$$undistr = [\langle \pi_1, i_1 \cdot \pi_2 \rangle, \langle \pi_1, i_2 \cdot \pi_2 \rangle]$$

$$\begin{tikzcd}
A \times (B + C) \arrow[rr, bend left, "distr"] \arrow[r, phantom, "\cong"] & {} & A \times B + A \times C \arrow[ll, bend left, "undistr"]
\end{tikzcd}$$

Distributividade

$$undistr = [\langle \pi_1, i_1 \cdot \pi_2 \rangle, \langle \pi_1, i_2 \cdot \pi_2 \rangle]$$

$$\begin{tikzcd} A \times (B + C) \arrow[r, bend left, "distr"] \arrow[r, phantom, "\cong"] & A \times B + A \times C \arrow[l, bend left, "undistr"] \end{tikzcd}$$

Pela definição de $f \times g$

$$undistr = [id \times i_1, id \times i_2]$$

Distributividade

$$undistr = [\langle \pi_1, i_1 \cdot \pi_2 \rangle, \langle \pi_1, i_2 \cdot \pi_2 \rangle]$$

$$\begin{tikzcd}
A \times (B + C) \arrow[r, bend left, "distr"] \arrow[r, phantom, "\cong"] & A \times B + A \times C \arrow[l, bend left, "undistr"]
\end{tikzcd}$$

Pela **lei da troca**

$$undistr = \langle [\pi_1, \pi_1], [i_1 \cdot \pi_2, i_2 \cdot \pi_2] \rangle$$

Em suma

Lei da troca:

$$\langle [f, g], [h, k] \rangle = [\langle f, h \rangle, \langle g, k \rangle]$$

"tenho um split e dava-me jeito uma alternativa..."

Em suma

Lei da troca:

$$\langle [f, g], [h, k] \rangle = [\langle f, h \rangle, \langle g, k \rangle]$$

"tenho um split e dava-me jeito uma alternativa..."

...e vice-versa

Cálculo
de
Programas

Aula T04

O tipo 1

Em Haskell:

$$() \;::\; ()$$

Logo

$$1 = \{()\}$$

Isomorfismo

$$\begin{tikzcd}
A \times 1 \arrow[r, bend left, "\pi_1"] \arrow[r, phantom, "\cong"] & A \arrow[l, bend left, "\langle id, \mathbf{!} \rangle"]
\end{tikzcd}$$

O tipo 1

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Isomorfismo

$$\begin{tikzcd}
A \times 1 \arrow[r, bend left, "\pi_1"] \arrow[r, phantom, "\cong"] & A \arrow[l, bend left, "\langle id, \mathbf{!} \rangle"]
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... "**!**" ?

O tipo 1

Facto funções envolvendo o tipo 1
são necessariamente **constantes**

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$$A \xrightarrow{!} 1$$

$$! = \underline{()}$$

O tipo 1

Facto funções envolvendo o tipo 1
são necessariamente **constantes**

"Bang":

$$A \xrightarrow{!} 1$$

$$! = \underline{()}$$

"Points":

$$1 \xrightarrow{\underline{a}} A$$

$$\underline{a}\,() = a$$

(desde que $a \in A$)

O tipo 0

$$0 = \{ \}$$

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Imediato em eg. Haskell:

```
data Zero
```

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Imediato em eg. Haskell:

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```

0 não é habitado: $\neg\,(a \in 0)$ para todo o a

Facto impossível $f : A \rightarrow 0$ quando $A \neq 0$

O tipo 0

$$\begin{tikzcd}
A \times 0 \arrow[rr, bend left, "id"] & = & 0 \arrow[ll, bend left, "id"]
\end{tikzcd}$$

O tipo 0

$$\begin{tikzcd}
A \times 0 \arrow[rr, bend left, "id"] \arrow[rr, phantom, "="] & & 0 \arrow[ll, bend left, "id"]
\end{tikzcd}$$

De facto

$$A \times 0 = \{(a, b) \mid a \in A \wedge b \in 0\}$$

O tipo 0

$$\begin{tikzcd}A \times 0 \arrow[rr, bend left, "id"] \arrow[rr, phantom, "="] & & 0 \arrow[ll, bend left, "id"]\end{tikzcd}$$

De facto

$$A \times 0 = \{(a, b) \mid a \in A \wedge b \in 0\}$$
$$A \times 0 = \{(a, b) \mid a \in A \wedge F\}$$

O tipo 0

$$\begin{tikzcd}
A \times 0 \arrow[rr, bend left, "id"] \arrow[r, phantom, ""] & = & 0 \arrow[ll, bend left, "id"]
\end{tikzcd}$$

De facto

$$A \times 0 = \{(a, b) \mid a \in A \wedge b \in 0\}$$

$$A \times 0 = \{(a, b) \mid a \in A \wedge F\}$$

$$A \times 0 = \{\,\}$$

O tipo $1 + A$

$$1 + A$$

O tipo $1 + A$

$$\begin{tikzcd}
1 \arrow[r, "i_1"] & 1 + A & A \arrow[l, "i_2"']
\end{tikzcd}$$

O tipo $1 + A$

$$\begin{tikzcd}
1 \arrow[r, "i_1"] & 1 + A & A \arrow[l, "i_2"'] \\
& B &
\end{tikzcd}$$

O tipo $1 + A$

$$\begin{tikzcd}
1 \arrow[r, "i_1"] & 1 + A & A \arrow[l, "i_2"'] \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

O tipo $1 + A$

$$\begin{tikzcd}
1 \arrow[r, "i_1"] \arrow[dr, "\underline{b_0}"'] & 1 + A & A \arrow[l, "i_2"'] \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

O tipo $1 + A$

$$\begin{tikzcd}
1 \arrow[r, "i_1"] \arrow[dr, "\underline{b_0}"'] & 1 + A \arrow[d, "{[\underline{b_0}, f]}" description] & A \arrow[l, "i_2"'] \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

O tipo $1 + A$

$$\begin{tikzcd}
1 \arrow[r, "i_1"] \arrow[dr, "\underline{b_0}"'] & 1 + A \arrow[d, "{[\underline{b_0}, f]}" description] & A \arrow[l, "i_2"'] \arrow[dl, "f"] \\
 & B &
\end{tikzcd}$$

$1 + A$

”apontador para A”

Mais isomorfismos!

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Voltando a

> *Obter uma morada de um funcionário público, sabendo que este se pode identificar através do seu número do cartão de cidadão (CC) ou do seu número de identificação fiscal (NIF).*

Mais isomorfismos!

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Em Haskell:

```
data Iden = CC Int | NIF Int
```

Mais isomorfismos!

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Em Haskell (Portugal):

```
data Iden = CC Int | NIF Int
```

Em Haskell (Alemanha):

```
data Iden = IDK Int | SZN Int
```

Mais isomorfismos!

Vamos ver isto no GHCI

Mais isomorfismos!

$$\begin{tikzcd}
Int \arrow[r, "i_1"] \arrow[dr, "\mathtt{CC}"'] & Int + Int \arrow[d, "{[\mathtt{CC}, \mathtt{NIF}]}" description] & Int \arrow[l, "i_2"'] \arrow[dl, "\mathtt{NIF}"] \\
& \mathtt{Iden} &
\end{tikzcd}$$

```
data Iden = CC Int | NIF Int
```

Mais isomorfismos!

$$\begin{tikzcd}
\mathtt{Iden} \arrow[r, bend left, "\mathbf{out} = \mathbf{in}^{\circ}"] \arrow[r, phantom, "\cong"] & Int + Int \arrow[l, bend left, "\mathbf{in} = {[\mathtt{CC}, \mathtt{NIF}]}"]
\end{tikzcd}$$

Mais isomorfismos!

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\end{tikzcd}$$

Sintaxe **concreta**

Mais isomorfismos!

$$\begin{tikzcd}
\mathtt{Iden} \arrow[rr, bend left, "\mathbf{out}=\mathbf{in}^{\circ}"] & \cong & Int + Int \arrow[ll, bend left, "\mathbf{in}=[\mathtt{CC},\mathtt{NIF}]"]
\end{tikzcd}$$

Sintaxe **concreta**

Sintaxe **abstracta**

Mais isomorfismos!

$$\begin{tikzcd}
\mathtt{Iden} \arrow[r, bend left, "\mathbf{out}=\mathbf{in}^{\circ}"] \arrow[r, phantom, "\cong"] & \mathit{Int}+\mathit{Int} \arrow[l, bend left, "\mathbf{in}=[\mathtt{CC},\mathtt{NIF}]"]
\end{tikzcd}$$

Sintaxe **concreta**

Sintaxe **abstracta**

in (entrar)

Mais isomorfismos!

$$\begin{tikzcd}
\mathtt{Iden} \arrow[r, bend left, "\mathbf{out}=\mathbf{in}^{\circ}"] \arrow[r, phantom, "\cong"] & \mathit{Int}+\mathit{Int} \arrow[l, bend left, "\mathbf{in}={[\mathtt{CC},\mathtt{NIF}]}"]
\end{tikzcd}$$

Sintaxe **concreta**

Sintaxe **abstracta**

in (entrar)

out (sair)

Maybe

$$\begin{tikzcd}
1 \arrow[r, "i_1"] \arrow[dr, swap, "\underline{\texttt{Nothing}}"] & 1 + A \arrow[d, "{[\underline{\texttt{Nothing}}, \texttt{Just}]}"] & A \arrow[l, swap, "i_2"] \arrow[dl, "\texttt{Just}"] \\
& \texttt{Maybe}\ A &
\end{tikzcd}$$

```
data Maybe a = Nothing | Just a
```

Maybe

$$\begin{tikzcd}
\texttt{Maybe}\ A \arrow[rr, bend left, "\mathbf{out}=\mathbf{in}^{\circ}"] & \cong & 1 + A \arrow[ll, bend left, "\mathbf{in} = {[}\underline{\texttt{Nothing}}{,}\texttt{Just}{]}"]
\end{tikzcd}$$

Maybe

$$\begin{tikzcd} \mathtt{Maybe}\ A \arrow[r, phantom, "\cong"] \arrow[r, bend left, "\mathbf{out}{=}\mathbf{in}^{\circ}"] & 1 + A \arrow[l, bend left, "{\mathbf{in} = [\underline{\mathtt{Nothing}}, \mathtt{Just}]}"] \end{tikzcd}$$

$$\mathbf{out} \cdot \mathbf{in} = id$$

Maybe

$$\mathbf{out} \cdot \mathbf{in} = \mathit{id}$$

Maybe

$$\begin{aligned} & \mathbf{out} \cdot \mathbf{in} = \mathit{id} \\ \Leftrightarrow \quad & \quad \{ \ \mathbf{in} = [\underline{\mathtt{Nothing}}, \mathtt{Just}] \ \} \\ & \mathbf{out} \cdot [\underline{\mathtt{Nothing}}, \mathtt{Just}] = \mathit{id} \end{aligned}$$

Maybe

$$
\begin{aligned}
& \mathsf{out} \cdot \mathsf{in} = id \\
\Leftrightarrow \quad & \{ \ \mathsf{in} = [\underline{\mathtt{Nothing}}, \mathtt{Just}] \ \} \\
& \mathsf{out} \cdot [\underline{\mathtt{Nothing}}, \mathtt{Just}] = id \\
\Leftrightarrow \quad & \{ \ \text{fusão-}+ \ \} \\
& [\mathsf{out} \cdot \underline{\mathtt{Nothing}}, \mathsf{out} \cdot \mathtt{Just}] = id
\end{aligned}
$$

Maybe

$$
\begin{array}{ll}
 & \mathbf{out} \cdot \mathbf{in} = id \\
\Leftrightarrow & \qquad \{ \ \mathbf{in} = [\underline{\mathtt{Nothing}}, \mathtt{Just}] \ \} \\
 & \mathbf{out} \cdot [\underline{\mathtt{Nothing}}, \mathtt{Just}] = id \\
\Leftrightarrow & \qquad \{ \ \text{fusão-}+ \ \} \\
 & [\mathbf{out} \cdot \underline{\mathtt{Nothing}}, \mathbf{out} \cdot \mathtt{Just}] = id \\
\Leftrightarrow & \qquad \{ \ \text{universal-}+\text{; função constante} \ \} \\
 & \left\{ \begin{array}{l} \underline{\mathbf{out} \cdot \mathtt{Nothing}} = i_1 \\ \mathbf{out} \cdot \mathtt{Just} = i_2 \end{array} \right.
\end{array}
$$

Maybe

$$\Leftrightarrow \qquad \{\ \text{variáveis}\ ;\ \underline{\texttt{Nothing}} : 1 \to \texttt{Maybe}\ A\ \}$$

$$\begin{cases} \underline{\mathbf{out} \cdot \texttt{Nothing}}\ () = i_1\ () \\ (\mathbf{out} \cdot \texttt{Just})\ a = i_2\ a \end{cases}$$

Maybe

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$$\Leftrightarrow \qquad \{\ \text{composição } (\textit{pointwise})\ \}$$

$$\begin{cases} \mathbf{out}\ \mathtt{Nothing} = i_1\ () \\ \mathbf{out}\ (\mathtt{Just}\ a) = i_2\ a \end{cases}$$

Maybe

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$$\left\{ \begin{array}{l} \mathbf{out}\ \mathtt{Nothing} = i_1\ () \\ \mathbf{out}\ (\mathtt{Just}\ a) = i_2\ a \end{array} \right.$$

$$\begin{tikzcd} \mathtt{Maybe}\ A \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & 1 + A \arrow[l, bend left, "\mathbf{in}\ =\ {[\underline{\mathtt{Nothing}}, \mathtt{Just}]}"] \end{tikzcd}$$

Either

$$\begin{tikzcd}
\texttt{Either}\ A\ B \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & A + B \arrow[l, bend left, "\mathbf{in} = {[\texttt{Left},\texttt{Right}]}"]
\end{tikzcd}$$

Either

$$\begin{tikzcd}
\mathtt{Either}\ A\ B \arrow[rr, bend left, "\mathsf{out}"] \arrow[rr, phantom, "\cong"] & & A + B \arrow[ll, bend left, "\mathsf{in} = {[\mathtt{Left},\mathtt{Right}]}"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "\mathtt{Left}"'] & A + B \arrow[d, "{[\mathtt{Left},\mathtt{Right}]}" description] & B \arrow[l, "i_2"'] \arrow[dl, "\mathtt{Right}"] \\
& \mathtt{Either}\ A\ B &
\end{tikzcd}$$

$[f, g]$

$\mathtt{either}\ f\ g$

Variantes de notação

$2 + 3$ notação **infixa** operador entre

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$2 + 3$	notação **infixa**	operador entre
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$+\ (2, 3)$	**prefixa** “uncurried”	argumentos agrupados

Variantes de notação

$f\ a\ b$ notação **prefixa** "curried" argumentos à vez

Variantes de notação

$f\ a\ b$	notação **prefixa** “curried”	argumentos à vez
$f\ (a, b)$	**prefixa** “uncurried”	argumentos agrupados

“Curried”? “uncurried”

Terminologia é homenagem ao matemático **Haskell Curry** (1900-1982) que desenvolveu a teoria de que deriva a **programação funcional** (1936).

Exemplo

$$\pi_1 : A \times B \to A$$

Exemplo

$$\pi_1 : A \times B \to A$$

$$\pi_1\left(a, b\right) = a$$

Exemplo

$$\pi_1 : A \times B \to A$$

$$\pi_1\,(a, b) = a$$

(notação prefixa "uncurried")

Exponenciação

```
Prelude> p1(a,b)=a
Prelude> p1’ a b = a
Prelude> :t p1
p1 :: (a, b) -> a
Prelude> :t p1’
p1’ :: a -> b -> a
Prelude>
```

Em Haskell

$$\pi_1{}' = \texttt{curry}\ \pi_1$$

Em Haskell

$$\pi_1{}' = \texttt{curry}\ \pi_1$$

$$\pi_1 = \texttt{uncurry}\ \pi_1{}'$$

Em Haskell

$$\pi_1{}' = \texttt{curry}\ \pi_1$$

$$\pi_1 = \texttt{uncurry}\ \pi_1{}'$$

Importante:

- `curry` (e `uncurry`) aceitam funções como **argumentos**
- `curry` (e `uncurry`) dão funções como **resultados**

Em Haskell

$$\pi_1{}' = \texttt{curry}\ \pi_1$$

$$\pi_1 = \texttt{uncurry}\ \pi_1{}'$$

Importante:

- `curry` (e `uncurry`) aceitam funções como **argumentos**
- `curry` (e `uncurry`) dão funções como **resultados**

Dizem-se funções de

ordem superior

Curry | uncurry

$$\texttt{curry} :: ((a, b) \to c) \to (a \to b \to c)$$

$$\texttt{uncurry} :: (a \to b \to c) \to (a, b) \to c$$

Curry | uncurry

$$\texttt{curry} :: \boxed{((a, b) \to c)} \to (a \to b \to c)$$

$$\texttt{uncurry} :: (a \to b \to c) \to \boxed{(a, b) \to c}$$

Curry | uncurry

$$\texttt{curry} :: \boxed{((a, b) \to c)} \to \boxed{(a \to b \to c)}$$

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Curry | uncurry

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inversas uma da outra?

Curry | uncurry

$$\texttt{curry} \;::\; ((a, b) \to c) \to a \to b \to c$$

Curry | uncurry

$\texttt{curry} :: ((a, b) \rightarrow c) \rightarrow a \rightarrow b \rightarrow c$

$\texttt{curry}$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \to c) \to a \to b \to c$$

$$\texttt{curry}\ f$$

Curry | uncurry

$$\texttt{curry} \; :: ((a, b) \rightarrow c) \rightarrow a \rightarrow b \rightarrow c$$

$$\texttt{curry} \; f \; a$$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \to c) \to a \to b \to c$$

$$\texttt{curry}\ f\ a\ b$$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \rightarrow c) \rightarrow a \rightarrow b \rightarrow c$$

$$\texttt{curry}\ f\ a\ b\ =$$

Curry | uncurry

$$\texttt{curry} \;::\; ((a, b) \rightarrow c) \rightarrow a \rightarrow b \rightarrow c$$

$$\texttt{curry}\; f\; a\; b \;=\; f\; (a, b)$$

Curry | uncurry

$$\texttt{curry} \ :: ((a, b) \to c) \to a \to b \to c$$

$$\texttt{curry} \ f \ a \ b \ = f \ (a, b)$$

$$\texttt{uncurry} \ :: (a \to b \to c) \to (a, b) \to c$$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \rightarrow c) \rightarrow a \rightarrow b \rightarrow c$$

$$\texttt{curry}\ f\ a\ b = f\ (a, b)$$

$$\texttt{uncurry} :: (a \rightarrow b \rightarrow c) \rightarrow (a, b) \rightarrow c$$

$$\texttt{uncurry}$$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \to c) \to a \to b \to c$$

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$$\texttt{uncurry}\ g$$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \to c) \to a \to b \to c$$

$$\texttt{curry}\ f\ a\ b = f\ (a, b)$$

$$\texttt{uncurry} :: (a \to b \to c) \to (a, b) \to c$$

$$\texttt{uncurry}\ g\ (a, b)$$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \to c) \to a \to b \to c$$

$$\texttt{curry}\ f\ a\ b = f\ (a, b)$$

$$\texttt{uncurry} :: (a \to b \to c) \to (a, b) \to c$$

$$\texttt{uncurry}\ g\ (a, b) =$$

Curry | uncurry

$$\texttt{curry} :: ((a, b) \rightarrow c) \rightarrow a \rightarrow b \rightarrow c$$

$$\texttt{curry}\ f\ a\ b = f\ (a, b)$$

$$\texttt{uncurry} :: (a \rightarrow b \rightarrow c) \rightarrow (a, b) \rightarrow c$$

$$\texttt{uncurry}\ g\ (a, b) = g\ a\ b$$

Curry | uncurry

$$\begin{cases} \texttt{curry}\ (\texttt{uncurry}\ f) = f? \\ \texttt{uncurry}\ (\texttt{curry}\ f) = f? \end{cases}$$

Curry | uncurry

$$\texttt{uncurry}\ g\ (a, b) = g\ a\ b$$

Curry | uncurry

$$\texttt{uncurry}\ (\texttt{curry}\ f)\ (a, b) = \texttt{curry}\ f\ a\ b$$

$$\texttt{curry}\ f\ a\ b = f\ (a, b)$$

Curry | uncurry

$$\texttt{uncurry}\ (\texttt{curry}\ f)\ (a, b) = f\ (a, b)$$

$$f\ x = g\ x \Leftrightarrow f = g$$

Curry | uncurry

$$\texttt{uncurry}\ (\texttt{curry}\ f) = f$$

$$f\ (g\ x) = (f \cdot g)\ x$$

Curry | uncurry

$$(\texttt{uncurry} \cdot \texttt{curry})\, f = f$$

$$id\ x = x$$

Curry | uncurry

$$(\texttt{uncurry} \cdot \texttt{curry})\ f = id\ f$$

$$f\ x = g\ x \Leftrightarrow f = g$$

Curry | uncurry

$$\texttt{uncurry} \cdot \texttt{curry} = \mathit{id}$$

□

Curry | uncurry

$$\texttt{curry}\ (\texttt{uncurry}\ f) = f\ ?$$

Curry | uncurry

$$\texttt{curry}\ f\ a\ b = f\ (a, b)$$

Curry | uncurry

$$\texttt{curry}\ (\texttt{uncurry}\ g)\ a\ b = \texttt{uncurry}\ g\ (a, b)$$

$$\texttt{uncurry}\ g\ (a, b) = g\ a\ b$$

Curry | uncurry

$$\texttt{curry}\ (\texttt{uncurry}\ g)\ a\ b = g\ a\ b$$

$$f\ x = g\ x \Leftrightarrow f = g$$

Curry | uncurry

$$\texttt{curry}\ (\texttt{uncurry}\ g)\ a = g\ a$$

$$f\ x = g\ x \Leftrightarrow f = g$$

Curry | uncurry

$$\texttt{curry}\ (\texttt{uncurry}\ g) = g$$

$$f\ (g\ x) = (f \cdot g)\ x$$

Curry | uncurry

$$(\texttt{curry}\ \cdot \texttt{uncurry}\ )\, g = g$$

$$id\ x = x$$

Curry | uncurry

$$(\texttt{curry} \cdot \texttt{uncurry})\ g = id\ g$$

$$f\ x = g\ x \Leftrightarrow f = g$$

Curry | uncurry

$$\texttt{curry} \cdot \texttt{uncurry} = \mathit{id}$$

□

Isomorfismo

$$\begin{tikzcd}
(a, b) \to c \arrow[r, bend left, "\texttt{curry}"] \arrow[r, phantom, "\cong"] & a \to b \to c \arrow[l, bend left, "\texttt{uncurry}"]
\end{tikzcd}$$

O tipo B^A

O tipo B^A

$$B^A = \{ f \mid f : A \to B \}$$

NB: as funções vão do expoente A para a base B

Isomorfismo

$$\begin{tikzcd} A \times B \to C \arrow[rr, bend left, "\texttt{curry}"] & \cong & A \to (B \to C) \arrow[ll, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

Isomorfismo

$$\begin{tikzcd} A \times B \to C \arrow[r, bend left, "\texttt{curry}"] \arrow[r, phantom, "\cong"] & A \to C^B \arrow[l, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

Isomorfismo

$$\begin{tikzcd}
C^{A \times B} \arrow[rr, bend left, "\texttt{curry}"] \arrow[rr, phantom, "\cong"] & & \left(C^{B}\right)^{A} \arrow[ll, bend left, "\texttt{uncurry}"]
\end{tikzcd}$$

Cf. potências numéricas: $c^{a\,b} = \left(c^{b}\right)^{a}$

Mais isomorfismos

$$a^1 = a \qquad 1^a = 1 \qquad a^0 = 1$$

$$\begin{tikzcd} A^1 \arrow[r, bend left, "g"] \arrow[r, phantom, "\cong"] & A \arrow[l, bend left, "f"] \end{tikzcd} \qquad \begin{tikzcd} 1^A \arrow[r, bend left, "g"] \arrow[r, phantom, "\cong"] & 1 \arrow[l, bend left, "f"] \end{tikzcd} \qquad \begin{tikzcd} A^0 \arrow[r, bend left, "g"] \arrow[r, phantom, "\cong"] & 1 \arrow[l, bend left, "f"] \end{tikzcd}$$

$$\begin{cases} f\ a = \underline{a} \\ g\ \underline{a} = a \end{cases} \qquad \begin{cases} f\ () = ! \\ g = ! \end{cases} \qquad \begin{cases} f\ () = \bot \\ g = ! \end{cases}$$

Curry | uncurry

$$\begin{tikzcd}
A \times B \to C \arrow[rr, bend left, "\texttt{curry}"] & \cong & A \to C^{B} \arrow[ll, bend left, "\texttt{uncurry}"]
\end{tikzcd}$$

Curry | uncurry

f

$$\begin{tikzcd} A \times B \to C \arrow[r, bend left, "\texttt{curry}"] \arrow[r, phantom, "\cong"] & A \to C^{B} \arrow[l, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

Curry | uncurry

$$f \xmapsto{\texttt{curry}} k$$

$$\begin{tikzcd}
A \times B \to C \arrow[rr, bend left, "\texttt{curry}"] & \cong & A \to C^{B} \arrow[ll, bend left, "\texttt{uncurry}"]
\end{tikzcd}$$

Curry | uncurry

$$f \xmapsto{\texttt{curry}} k$$

$$\begin{tikzcd} A \times B \to C \arrow[r, phantom, "\cong"] \arrow[r, bend left, "\texttt{curry}"] & A \to C^{B} \arrow[l, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

k

Curry | uncurry

$$\begin{tikzcd}
f \arrow[r, mapsto, "\texttt{curry}"] & k
\end{tikzcd}$$

$$\begin{tikzcd}
A \times B \to C \arrow[rr, bend left, "\texttt{curry}"] & \cong & A \to C^B \arrow[ll, bend left, "\texttt{uncurry}"]
\end{tikzcd}$$

$$\begin{tikzcd}
f & k \arrow[l, mapsto, "\texttt{uncurry}"']
\end{tikzcd}$$

Exponenciação

$$\begin{tikzcd} & C \\ A \times B \arrow[ur, "f"'] & \end{tikzcd}$$

Exponenciação

$$\begin{tikzcd}
C^B & & C \\
A \arrow[u, "\overline{f}"] & A \times B \arrow[ur, "f"'] &
\end{tikzcd}$$

$\overline{f}$ abrevia `curry` f

Exponenciação

$$\begin{tikzcd}
C^B \\
A \arrow[u, "\overline{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B & C \\
A \times B \arrow[u, "\overline{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$\overline{f}$ abrevia `curry` f

Exponenciação

$$\begin{tikzcd}
C^B \\
A \arrow[u, "\overline{f}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C^B & {} & B & C \\
A \arrow[u, "\overline{f}"] & \times & B \arrow[u, "id"'] \arrow[ur, "f"']
\end{tikzcd}$$

$\overline{f}$ abrevia `curry` f

Exponenciação

$$\begin{tikzcd}
C^B & C^B \times B & C \\
A \arrow[u, "\overline{f}"] & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$\overline{f}$ abrevia `curry` f

De volta a $\underbrace{\texttt{curry}\ f}_{\overline{f}}\ a\ b = f\ (a, b)$

De volta a $\underbrace{\texttt{curry}\ f}_{\overline{f}}\ a\ b = f\ (a, b)$

$$\overbrace{\overline{f}\ a}^{g}\ b = f\ (a, b)$$

De volta a $\underbrace{\texttt{curry}\ f}_{\overline{f}}\ a\ b = f\ (a, b)$

$$\overbrace{\overline{f}\ a}^{g}\ b = f\ (a, b)$$

$$\Leftrightarrow \qquad \{\ \text{introduzir } g = \overline{f}\ a\ \}$$

$$g\ b = f\ (a, b)$$

De volta a $\underbrace{\texttt{curry}\ f}_{\overline{f}}\ a\ b = f\ (a, b)$

$$\begin{array}{ll}
 & \overbrace{\overline{f}\ a}^{g}\ b = f\ (a, b) \\
\Leftrightarrow & \qquad \{\ \text{introduzir } g = \overline{f}\ a\ \} \\
 & g\ b\ =\ f\ (a, b) \\
\Leftrightarrow & \qquad \{\ \text{introduzir } ap\ (f, x) = f\ x\ \} \\
 & ap\ (g, b)\ =\ f\ (a, b)
\end{array}$$

Vamos em $ap\ (g, b) = f\ (a, b)$

$$
\begin{array}{ll}
\Leftrightarrow & \quad \{\ g = \overline{f}\ a\ ;\ \text{natural-}id\ \} \\
& ap\ (\overline{f}\ a, id\ b) \ = \ f\ (a, b)
\end{array}
$$

Vamos em $ap\ (g, b) = f\ (a, b)$

$$
\begin{array}{ll}
\Leftrightarrow & \quad \{\ g = \overline{f}\ a\ ;\ \text{natural-}id\ \} \\
& ap\ (\overline{f}\ a, id\ b)\ =\ f\ (a, b) \\
\Leftrightarrow & \quad \{\ \text{produto}\ (f \times g)\ (x, y) = (f\ x, g\ y)\ \} \\
& ap\ ((\overline{f} \times id)\ (a, b))\ =\ f\ (a, b)
\end{array}
$$

Vamos em $ap\ (g, b) = f\ (a, b)$

$$
\begin{array}{cl}
 & ap\ (g, b) = f\ (a, b) \\
\Leftrightarrow & \quad \{\ g = \overline{f}\ a\ ;\ \text{natural-}id\ \} \\
 & ap\ (\overline{f}\ a, id\ b) \ = \ f\ (a, b) \\
\Leftrightarrow & \quad \{\ \text{produto}\ (f \times g)\ (x, y) = (f\ x, g\ y)\ \} \\
 & ap\ ((\overline{f} \times id)\ (a, b)) \ = \ f\ (a, b) \\
\Leftrightarrow & \quad \{\ \text{composição}\ \} \\
 & (ap \cdot (\overline{f} \times id))\ (a, b) \ = \ f\ (a, b)
\end{array}
$$

Vamos em $ap\ (g, b) = f\ (a, b)$

$$\Leftrightarrow \quad \{\ g = \overline{f}\ a\ ;\ \text{natural-}id\ \}$$

$$ap\ (\overline{f}\ a, id\ b) \ = \ f\ (a, b)$$

$$\Leftrightarrow \quad \{\ \text{produto}\ (f \times g)\ (x, y) = (f\ x, g\ y)\ \}$$

$$ap\ ((\overline{f} \times id)\ (a, b)) \ = \ f\ (a, b)$$

$$\Leftrightarrow \quad \{\ \text{composição}\ \}$$

$$(ap \cdot (\overline{f} \times id))\ (a, b) \ = \ f\ (a, b)$$

$$\Leftrightarrow \quad \{\ \text{eliminar variáveis } a \text{ and } b\ \}$$

$$ap \cdot (\overline{f} \times id) \ = \ f$$

Cancelamento

$$\begin{tikzcd}
C^B & C^B \times B & C \\
A \arrow[u, "\overline{f}"] & A \times B \arrow[u, "\overline{f} \times id" description] \arrow[ur, "f"'] &
\end{tikzcd}$$

Cancelamento

$$\begin{tikzcd}
C^B & & C^B \times B \arrow[r, "ap"] & C \\
A \arrow[u, "\overline{f}"] & & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$$ap \cdot (\overline{f} \times id) = f$$

Cancelamento

$$\begin{tikzcd}
C^B \\
A \arrow[u, "k = \overline{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[r, "ap"] & C \\
A \times B \arrow[u, "k \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$$k = \overline{f} \Rightarrow ap \cdot (k \times id) = f$$

Propriedade universal

$$\begin{tikzcd}
C^B & & C^B \times B \arrow[r, "ap"] & C \\
A \arrow[u, "k = \overline{f}"] & & A \times B \arrow[u, "k \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$$k = \overline{f} \Leftrightarrow ap \cdot (k \times id) = f$$

Reflexão ($k := id$)

$$\begin{tikzcd}
C^B \\
C^B \arrow[u, "id = \overline{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[r, "ap"] & C \\
C^B \times B \arrow[u, "k \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$$id = \overline{f} \Leftrightarrow ap = f$$

Fusão

$$\begin{tikzcd}
C^B \\
A \arrow[u, "\overline{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[r, "ap"] & C \\
A \times B \arrow[u, "\overline{f} \times id" description] \arrow[ur, "f"']
\end{tikzcd}$$

Fusão

$$\begin{tikzcd}
C^B \\
A \arrow[u, "\overline{f}"] \\
D \arrow[u, "g"]
\end{tikzcd}$$

$$\begin{tikzcd}
C^B \times B \arrow[r, "ap"] & C \\
A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Fusão

$$\begin{tikzcd}
C^B \\
A \arrow[u, "\overline{f}"] \\
D \arrow[u, "g"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[r, "ap"] & C \\
A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] & \\
D \times B \arrow[u, "g \times id"] &
\end{tikzcd}$$

Fusão

$$\begin{tikzcd}
C^B & & C^B \times B \arrow[r, "ap"] & C \\
A \arrow[u, "\overline{f}"] & & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] & \\
D \arrow[u, "g"] & & D \times B \arrow[u, "g \times id"] \arrow[uur, "f \cdot (g \times id)"'] &
\end{tikzcd}$$

Fusão

$$\begin{tikzcd}
C^B & & C^B \times B \arrow[r, "ap"] & C \\
& & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] & \\
D \arrow[uu, "\overline{f} \cdot g"] & & D \times B \arrow[u, "g \times id"] \arrow[uur, "f \cdot (g \times id)"'] &
\end{tikzcd}$$

Fusão

$$\begin{tikzcd}
C^B & C^B \times B \arrow[r, "ap"] & C \\
 & A \times B \arrow[u, "\overline{f} \times id"'] \arrow[ur, "f"'] & \\
D \arrow[uu, "\overline{f \cdot (g \times id)}"] & D \times B \arrow[u, "g \times id"'] \arrow[uur, "f \cdot (g \times id)"'] &
\end{tikzcd}$$

$$\overline{f} \cdot g = \overline{f \cdot (g \times id)}$$

Curry | uncurry

$$\begin{tikzcd} f \arrow[rr, mapsto, "\texttt{curry}"] & & k \end{tikzcd}$$

$$\begin{tikzcd} A \times B \to C \arrow[r, phantom, "\cong"] \arrow[rr, bend left, "\texttt{curry}"] & & A \to C^{B} \arrow[ll, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

$$\begin{tikzcd} f & & k \arrow[ll, mapsto, "\texttt{uncurry}"'] \end{tikzcd}$$

Curry | uncurry

$$\begin{tikzcd} f \arrow[rr, mapsto, "\texttt{curry}"] & & k \end{tikzcd}$$

$$\begin{tikzcd} A \times B \to C \arrow[r, phantom, "\cong"] \arrow[rr, bend left, "\texttt{curry}"] & & A \to C^B \arrow[ll, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

$$\begin{tikzcd} f & & k \arrow[ll, mapsto, "\texttt{uncurry}"'] \end{tikzcd}$$

Abreviaturas:

$$\texttt{curry}\ f = \overline{f}$$

Curry | uncurry

$$\begin{tikzcd} f \arrow[rrr, mapsto, "\texttt{curry}"] & & & k \end{tikzcd}$$

$$\begin{tikzcd} A \times B \to C \arrow[r, bend left, "\texttt{curry}"] \arrow[r, phantom, "\cong"] & A \to C^B \arrow[l, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

$$\begin{tikzcd} f & & & k \arrow[lll, mapsto, "\texttt{uncurry}"'] \end{tikzcd}$$

Abreviaturas:

$$\texttt{curry}\ f = \overline{f}$$

$$\texttt{uncurry}\ f = \widehat{f}$$

Curry | uncurry

$$\begin{tikzcd} f \arrow[r, mapsto, "\texttt{curry}"] & k \end{tikzcd}$$

$$\begin{tikzcd} A \times B \to C \arrow[r, phantom, "\cong"] \arrow[r, bend left, "\texttt{curry}"] & A \to C^B \arrow[l, bend left, "\texttt{uncurry}"] \end{tikzcd}$$

$$\begin{tikzcd} f & k \arrow[l, mapsto, "\texttt{uncurry}"'] \end{tikzcd}$$

Abreviaturas:

$$\texttt{curry}\ f = \overline{f}$$

$$\texttt{uncurry}\ f = \widehat{f}$$

Isomorfismos:

$$f = g \Leftrightarrow \overline{f} = \overline{g}$$

$$k = h \Leftrightarrow \widehat{k} = \widehat{h}$$

Absorção (exponenciação)

$$\begin{tikzcd}
C^B & C^B \times B \arrow[r, "ap"] & C \\
A \arrow[u, "\overline{f}"] & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Absorção (exponenciação)

$$\begin{tikzcd}
C^B \\
A \arrow[u, "\overline{f}"]
\end{tikzcd}$$

$$\begin{tikzcd}
& D \\
C^B \times B \arrow[r, "ap"] & C \arrow[u, "g"'] \\
A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Absorção (exponenciação)

$$\begin{tikzcd}
 & & D \\
C^B & C^B \times B \arrow[r, "ap"] \arrow[ur, "g \cdot ap"] & C \arrow[u, "g"'] \\
A \arrow[u, "\overline{f}"] & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Absorção (exponenciação)

$$\begin{tikzcd}
D^B & & D \\
C^B \arrow[u, "\overline{g \cdot ap}"] & C^B \times B \arrow[r, "ap"] \arrow[ur, "g \cdot ap"] & C \arrow[u, "g"'] \\
A \arrow[u, "\overline{f}"] & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Absorção (exponenciação)

$$\begin{tikzcd}
D^B & & D^B \times B & & D \\
C^B \arrow[u, "\overline{g \cdot ap}"] & & C^B \times B \arrow[u, "\overline{g \cdot ap} \times id"] \arrow[rru, "g \cdot ap"] \arrow[rr, "ap"] & & C \arrow[u, "g"'] \\
A \arrow[u, "\overline{f}"] & & A \times B \arrow[u, "\overline{f} \times id"] \arrow[rru, "f"'] & &
\end{tikzcd}$$

Absorção (exponenciação)

$$\begin{tikzcd}
D^B & & D^B \times B \arrow[r, "ap"] & D \\
C^B \arrow[u, "\overline{g \cdot ap}"] & & C^B \times B \arrow[u, "\overline{g \cdot ap} \times id"] \arrow[r, "ap"] \arrow[ur, "g \cdot ap"] & C \arrow[u, "g"'] \\
A \arrow[u, "\overline{f}"] & & A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Absorção

$$\begin{tikzcd}
D^B \\
C^B \arrow[u, "g^B"] \\
A \arrow[u, "\overline{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
D^B \times B \arrow[r, "ap"] & D \\
C^B \times B \arrow[u, "g^B \times id"] \arrow[r, "ap"] \arrow[ur, "g \cdot ap"] & C \arrow[u, "g"'] \\
A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$$g^B = \overline{g \cdot ap}$$

Absorção

$$\begin{tikzcd}
D^B \\
C^B \arrow[u, "g^B"] \\
A \arrow[u, "\overline{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
D^B \times B \arrow[r, "ap"] & D \\
C^B \times B \arrow[u, "g^B \times id"] \arrow[ur, "g \cdot ap"] \arrow[r, "ap"] & C \arrow[u, "g"'] \\
A \times B \arrow[u, "\overline{f} \times id"] \arrow[ur, "f"']
\end{tikzcd}$$

$$exp\ g = \overline{g \cdot ap}$$

Absorção

$$\begin{tikzcd}
D^B \\
A \arrow[u, "g^B \cdot \overline{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
D^B \times B \arrow[r, "ap"] & D \\
A \times B \arrow[u, "g^B \cdot \overline{f} \times id"] \arrow[ur, "g \cdot f"']
\end{tikzcd}$$

Absorção

$$\begin{tikzcd}
D^B \\
A \arrow[u, "\overline{g \cdot f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
D^B \times B \arrow[r, "ap"] & D \\
A \times B \arrow[u, "\overline{g \cdot f} \times id"] \arrow[ur, "g \cdot f"'] &
\end{tikzcd}$$

$$\overline{g \cdot f} = g^B \cdot \overline{f}$$

Sumário

$f \cdot g$ sequencial

Sumário

$f \cdot g$ sequencial

$\langle f , g \rangle$

Sumário

$f \cdot g$	sequencial
$\langle f, g \rangle$, $f \times g$	paralela

Sumário

$f \cdot g$	sequencial
$\langle f , g \rangle , \; f \times g$	paralela
$[f , g]$	

Sumário

$f \cdot g$ sequencial

$\langle f , g \rangle$, $f \times g$ paralela

$[f , g]$, $f + g$

Sumário

$f \cdot g$	sequencial
$\langle f , g \rangle$, $f \times g$	paralela
$[f , g]$, $f + g$, $p \rightarrow f \ , g$	alternativa

Sumário

$f \cdot g$	sequencial
$\langle f , g \rangle$, $f \times g$	paralela
$[f , g]$, $f + g$, $p \rightarrow f \, , \, g$	alternativa
$\overline{f}$	

Sumário

$f \cdot g$	sequencial
$\langle f , g \rangle$, $f \times g$	paralela
$[f , g]$, $f + g$, $p \rightarrow f \; , g$	alternativa
$\overline{f}$, $exp\ f$	ordem superior

Sumário

$f \cdot g$	sequencial
$\langle f , g \rangle$, $f \times g$	paralela
$[f , g]$, $f + g$, $p \rightarrow f \, , \, g$	alternativa
$\overline{f}$, $exp\ f$	ordem superior

Programação composicional 🙂

Sumário

$f \cdot g$	sequencial
$\langle f , g \rangle$, $f \times g$	paralela
$[f , g]$, $f + g$, $p \rightarrow f , g$	alternativa
$\overline{f}$, $\mathit{exp}\ f$	ordem superior
	Programação composicional ☺
?	recursiva ?

Sumário

$A \times B$ registos (“structs”)

Sumário

$A \times B$ registos ("structs")

$A + B$ registos variantes ("unions")

Sumário

$A \times B$	registos (“structs”)
$A + B$	registos variantes (“unions”)
$1 + A$	“apontadores”

Sumário

$A \times B$	registos (“structs”)
$A + B$	registos variantes (“unions”)
$1 + A$	“apontadores”
B^A	“arrays” , “streams”

Sumário

$A \times B$	registos (“structs”)
$A + B$	registos variantes (“unions”)
$1 + A$	“apontadores”
B^A	“arrays”, “streams”

estruturação de dados ☺

Sumário

$A \times B$	registos ("structs")
$A + B$	registos variantes ("unions")
$1 + A$	"apontadores"
B^A	"arrays", "streams"
	estruturação de dados 🙂
$1 + A \times X^2$	estruturas polinomiais

Sintaxes concretas

Sintaxe abstracta	Pascal	C/C++	**Description**
$A \times B$			Records
$A + B$			Variant records
B^A	`array[A] of B`	`B ...[A]`	Arrays
$1 + A$	`^A`	`A *...`	Pointers

$A \times B$

```
record
   P: A;
   S: B
end;
```

```
struct {
   A first;
   B second;
};
```

$A + B$

```
record
   case
    tag:  integer
        of x =
     1:  (P:A);
     2:  (S:B)
end;
```

```
struct {
   int tag; /* 1,2 */
   union {
      A ifA;
      B ifB;
   } data;
};
```

Cálculo
de
Programas

Aula T05

Polimorfismo

$$reverse :: [a] \to [a]$$

Polimorfismo

$$reverse :: [a] \to [a]$$

$$reverse :: [Int] \to [Int]$$

Polimorfismo

$$reverse :: [a] \rightarrow [a]$$

$$reverse :: [Int] \rightarrow [Int]$$
$$reverse :: String \rightarrow String$$

Polimorfismo

$$reverse :: [a] \to [a]$$

$$reverse :: [Int] \to [Int]$$
$$reverse :: String \to String$$
$$\ldots$$

Polimorfismo

$$reverse :: [a] \to [a]$$

$$reverse :: [Int] \to [Int]$$
$$reverse :: String \to String$$
$$\dots$$

"From the **type** of a **polymorphic function** we can derive a **theorem** that it satisfies. (...) How useful are the theorems so generated? Only time and experience will tell (...)" [**Philip Wadler** 1989]

Propriedades grátis

$$[2, 5, 0, 5, 1]$$

Propriedades grátis

$$\begin{tikzcd}[1, 5, 0, 5, 2] & [2, 5, 0, 5, 1] \arrow[l, mapsto, "\textit{reverse}"']\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[1, 5, 0, 5, 2]} \arrow[d, mapsto, "succs"'] & {[2, 5, 0, 5, 1]} \arrow[l, mapsto, "reverse"'] \\
{[2, 6, 1, 6, 3]} &
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[1, 5, 0, 5, 2]} \arrow[d, mapsto, "\mathit{succs}"'] & {[2, 5, 0, 5, 1]} \arrow[l, mapsto, "\mathit{reverse}"'] \arrow[d, mapsto, "\mathit{succs}"] \\
{[2, 6, 1, 6, 3]} & {[3, 6, 1, 6, 2]}
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[1, 5, 0, 5, 2]} \arrow[d, mapsto, "\mathit{succs}"'] & & {[2, 5, 0, 5, 1]} \arrow[ll, mapsto, "\mathit{reverse}"'] \arrow[d, mapsto, "\mathit{succs}"] \\
{[2, 6, 1, 6, 3]} & & {[3, 6, 1, 6, 2]} \arrow[ll, mapsto, "\mathit{reverse}"]
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[1,5,0,5,2]} \arrow[d, mapsto, "\mathit{map}\ \mathit{succ}"'] & {[2,5,0,5,1]} \arrow[l, mapsto, "\mathit{reverse}"'] \arrow[d, mapsto, "\mathit{map}\ \mathit{succ}"] \\
{[2,6,1,6,3]} & {[3,6,1,6,2]} \arrow[l, mapsto, "\mathit{reverse}"]
\end{tikzcd}$$

Propriedades grátis

$$[1, 5, 0, 5, 2] \xmapsfrom{\mathit{reverse}} [2, 5, 0, 5, 1]$$

Propriedades grátis

$$\begin{tikzcd}
{[1, 5, 0, 5, 2]} \arrow[d, mapsto, "\mathit{map}\ \mathit{even}"'] & & {[2, 5, 0, 5, 1]} \arrow[ll, mapsto, "\mathit{reverse}"'] \\
{[F, F, T, F, T]} & &
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[1, 5, 0, 5, 2]} \arrow[d, mapsto, "map\ even"'] & {[2, 5, 0, 5, 1]} \arrow[l, mapsto, "reverse"'] \arrow[d, mapsto, "map\ even"] \\
{[F, F, T, F, T]} & {[T, F, T, F, F]}
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[1, 5, 0, 5, 2]} \arrow[d, mapsto, "\mathit{map}\ \mathit{even}"'] & {[2, 5, 0, 5, 1]} \arrow[l, mapsto, "\mathit{reverse}"'] \arrow[d, mapsto, "\mathit{map}\ \mathit{even}"] \\
{[F, F, T, F, T]} & {[T, F, T, F, F]} \arrow[l, mapsto, "\mathit{reverse}"]
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}[2,5,0] & [2,5,0,5,1] \arrow[l, mapsto, "\mathit{take}\ 3"']\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[2, 5, 0]} \arrow[d, mapsto, "\mathit{map}\ (2*)"'] & & {[2, 5, 0, 5, 1]} \arrow[ll, mapsto, "\mathit{take}\ 3"'] \\
{[4, 10, 0]} & &
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[2, 5, 0]} \arrow[d, mapsto, "\mathit{map}\ (2*)"'] & & {[2, 5, 0, 5, 1]} \arrow[ll, mapsto, "\mathit{take}\ 3"'] \arrow[d, mapsto, "\mathit{map}\ (2*)"] \\
{[4, 10, 0]} & & {[4, 10, 0, 10, 2]}
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
{[2, 5, 0]} \arrow[d, mapsto, "\mathit{map}\ (2*)"'] & {[2, 5, 0, 5, 1]} \arrow[l, mapsto, "\mathit{take}\ 3"'] \arrow[d, mapsto, "\mathit{map}\ (2*)"] \\
{[4, 10, 0]} & {[4, 10, 0, 10, 2]} \arrow[l, mapsto, "\mathit{take}\ 3"]
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}\texttt{"ogramas"} & \texttt{"Programas"} \arrow[l, mapsto, "\mathit{drop}\ 2"']\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
\texttt{"ogramas"} \arrow[d, mapsto, "\mathit{map}\ \mathit{upper}"'] & \texttt{"Programas"} \arrow[l, mapsto, "\mathit{drop}\ 2"'] \\
\texttt{"OGRAMAS"} &
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
\texttt{"ogramas"} \arrow[d, mapsto, "\mathit{map}\ \mathit{upper}"'] & \texttt{"Programas"} \arrow[l, mapsto, "\mathit{drop}\ 2"'] \arrow[d, mapsto, "\mathit{map}\ \mathit{upper}"] \\
\texttt{"OGRAMAS"} & \texttt{"PROGRAMAS"}
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
\texttt{"ogramas"} \arrow[d, mapsto, "\mathit{map}\ \mathit{upper}"'] & \texttt{"Programas"} \arrow[l, mapsto, "\mathit{drop}\ 2"'] \arrow[d, mapsto, "\mathit{map}\ \mathit{upper}"] \\
\texttt{"OGRAMAS"} & \texttt{"PROGRAMAS"} \arrow[l, mapsto, "\mathit{drop}\ 2"]
\end{tikzcd}$$

De volta a...

$$\begin{tikzcd}(A \times B) \times C \arrow[rr, bend left, "assocr"] \arrow[r, phantom, "\cong"] & {} & A \times (B \times C) \arrow[ll, bend left, "assocl"]\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Propriedade grátis

$$((2, T), \texttt{"CP"}) \xleftarrow{\mathit{assocl}} (2, (T, \texttt{"CP"}))$$

Propriedade grátis

$$\begin{tikzcd}
((2, T), \texttt{"CP"}) \arrow[d, mapsto, "(succ \times \neg) \times !"'] & (2, (T, \texttt{"CP"})) \arrow[l, mapsto, "assocl"'] \\
((3, F), ()) &
\end{tikzcd}$$

Propriedade grátis

$$\begin{tikzcd}
((2, T), \texttt{"CP"}) \arrow[d, mapsto, "(\mathit{succ} \times \neg) \times !"'] & (2, (T, \texttt{"CP"})) \arrow[l, mapsto, "\mathit{assocl}"'] \arrow[d, mapsto, "\mathit{succ} \times (\neg \times !)"] \\
((3, F), ()) & (3, (F, ()))
\end{tikzcd}$$

Propriedade grátis

$$\begin{tikzcd}
((2, T), \texttt{"CP"}) \arrow[d, mapsto, "(\mathit{succ} \times \neg) \times !"'] & (2, (T, \texttt{"CP"})) \arrow[l, mapsto, "\mathit{assocl}"'] \arrow[d, mapsto, "\mathit{succ} \times (\neg \times !)"] \\
((3, F), ()) & (3, (F, ())) \arrow[l, mapsto, "\mathit{assocl}"]
\end{tikzcd}$$

Propriedade grátis

$$\begin{tikzcd}((a, b), c) & (a, (b, c)) \arrow[l, mapsto, "assocl"']\end{tikzcd}$$

Propriedade grátis

$$\begin{tikzcd}
((a, b), c) \arrow[d, mapsto, "(f \times g) \times h"'] & (a, (b, c)) \arrow[l, mapsto, "assocl"'] \\
((f\ a, g\ b), h\ c) &
\end{tikzcd}$$

Propriedade grátis

$$\begin{tikzcd}
((a, b), c) \arrow[d, mapsto, "(f \times g) \times h"'] & & (a, (b, c)) \arrow[ll, mapsto, "assocl"'] \arrow[d, mapsto, "f \times (g \times h)"] \\
((f\ a, g\ b), h\ c) & & (f\ a, (g\ b, h\ c))
\end{tikzcd}$$

Propriedade grátis

$$\begin{tikzcd}
((a, b), c) \arrow[d, mapsto, "(f \times g) \times h"'] & & (a, (b, c)) \arrow[ll, mapsto, "assocl"'] \arrow[d, mapsto, "f \times (g \times h)"] \\
((f\ a, g\ b), h\ c) & & (f\ a, (g\ b, h\ c)) \arrow[ll, mapsto, "assocl"]
\end{tikzcd}$$

Propriedades grátis

$$A \times (B \times C)$$

Propriedades grátis

$$\begin{tikzcd}(A \times B) \times C & A \times (B \times C) \arrow[l, "assocl"']\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
(A \times B) \times C \arrow[d, "(f \times g) \times h"'] & A \times (B \times C) \arrow[l, "\mathit{assocl}"'] \\
(D \times E) \times F &
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
(A \times B) \times C \arrow[d, "(f \times g) \times h"'] & A \times (B \times C) \arrow[l, "assocl"'] \arrow[d, "f \times (g \times h)"] \\
(D \times E) \times F & D \times (E \times F)
\end{tikzcd}$$

Propriedades grátis

$$\begin{tikzcd}
(A \times B) \times C \arrow[d, "(f \times g) \times h"'] & & A \times (B \times C) \arrow[ll, "assocl"'] \arrow[d, "f \times (g \times h)"] \\
(D \times E) \times F & & D \times (E \times F) \arrow[ll, "assocl"]
\end{tikzcd}$$

$$((f \times g) \times h) \cdot assocl = assocl \cdot (f \times (g \times h))$$

De volta a...

$$\begin{tikzcd}
A \times B \arrow[r, bend left, "swap"] \arrow[r, phantom, "\cong"] & B \times A \arrow[l, bend left, "swap"]
\end{tikzcd}$$

$$swap \cdot swap = id$$

Regra prática

$$\begin{tikzcd} B \times A & A \times B \arrow[l, "swap"'] \end{tikzcd}$$

Regra prática

$$\begin{tikzcd}
B \times A & A \times B \arrow[l, "swap"'] \\
\\
B' \times A' & A' \times B' \arrow[l, "swap"]
\end{tikzcd}$$

Regra prática

$$\begin{tikzcd}
B \times A \arrow[d, "{''B \times A''}"'] & A \times B \arrow[l, "swap"'] \arrow[d, "{''A \times B''}"] \\
B' \times A' & A' \times B' \arrow[l, "swap"]
\end{tikzcd}$$

Regra prática

$$\begin{tikzcd}
B \times A \arrow[d, "g \times f"'] & & A \times B \arrow[ll, "swap"'] \arrow[d, "f \times g"] \\
B' \times A' & & A' \times B' \arrow[ll, "swap"]
\end{tikzcd}$$

$$(g \times f) \cdot swap = swap \cdot (f \times g)$$

Regra prática

Nas setas verticais, que têm aspas

- Substituir $A := f$, $B := g$, etc

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Nas setas verticais, que têm aspas

- Substituir $A := f$, $B := g$, etc
- No caso de tipos concretos, substituir por id, eg. $2 := id$

Regra prática

Nas setas verticais, que têm aspas

- Substituir $A := f$, $B := g$, etc
- No caso de tipos concretos, substituir por id, eg. $2 := id$
- Retirar as aspas.

Caso mais simples...

$$\begin{tikzcd}A & & A \arrow[ll, "id"']\end{tikzcd}$$

Caso mais simples...

$$\begin{tikzcd}
A & A \arrow[l, "id"'] \\
A' & A' \arrow[l, "id"]
\end{tikzcd}$$

Caso mais simples...

$$\begin{tikzcd}
A \arrow[d, "''A''"'] & A \arrow[l, "id"'] \arrow[d, "''A''"] \\
A' & A' \arrow[l, "id"]
\end{tikzcd}$$

Natural-*id*

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \\
A' & A' \arrow[l, "id"]
\end{tikzcd}$$

$$f \cdot id = id \cdot f$$

Natural-π_1

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_1"'] \end{tikzcd}$$

Natural-π_1

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \\
A' & A' \times B' \arrow[l, "\pi_1"]
\end{tikzcd}$$

Natural-π_1

$$\begin{tikzcd}
A \arrow[d, "{''A''}"'] & A \times B \arrow[l, "\pi_1"'] \arrow[d, "{''A \times B''}"] \\
A' & A' \times B' \arrow[l, "\pi_1"]
\end{tikzcd}$$

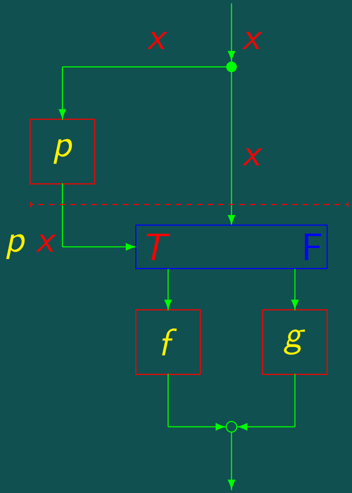
Natural-π_1

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "\pi_1"'] \arrow[d, "f \times g"] \\
A' & A' \times B' \arrow[l, "\pi_1"]
\end{tikzcd}$$

$$(f) \cdot \pi_1 = \pi_1 \cdot (f \times g)$$

De volta ao "if-then-else"

$$\begin{tikzcd}
 & A \arrow[d, "\langle p, id \rangle"] & \\
 & 2 \times A \arrow[d, "\alpha^\circ"] & \\
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + A & A \arrow[l, "i_2"'] \arrow[dl, "g"] \\
 & B &
\end{tikzcd}$$



2ª lei de fusão do condicional

Aulas TP:

$$(p \rightarrow f \; , g) \cdot h \;=\; p \cdot h \rightarrow f \cdot h \, , g \cdot h$$

2ª lei de fusão do condicional

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$$(p \rightarrow f\ , g) \cdot h \;=\; p \cdot h \rightarrow f \cdot h\ , g \cdot h$$

consequência de

$$p? \cdot f \;=\; (f + f) \cdot (p \cdot f)?$$

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Aulas TP:

$$(p \rightarrow f \ , g) \cdot h \ = \ p \cdot h \rightarrow f \cdot h \, , g \cdot h$$

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E **prova** desta última?

2ª lei de fusão do condicional

Aulas TP:

$$(p \rightarrow f \;, g) \cdot h \;=\; p \cdot h \rightarrow f \cdot h \;, g \cdot h$$

consequência de

$$p? \cdot f \;=\; (f + f) \cdot (p \cdot f)?$$

E **prova** desta última?

("Só mais tarde...")

Vamos provar agora

Provar

$$p? \cdot f \; = \; (f + f) \cdot (p \cdot f)?$$

sabendo:

$$\begin{tikzcd}
A \arrow[r, "\langle p, id \rangle"] \arrow[rr, bend right, "p?"'] & 2 \times A \arrow[r, "\alpha^\circ"] & A + A
\end{tikzcd}$$

Relembrar

$$\begin{tikzcd}
2 \times A \arrow[rr, bend left, "\alpha^{\circ}"] \arrow[rr, phantom, "\cong"] & & A + A \arrow[ll, bend left, "\alpha"]
\end{tikzcd}$$

$$\alpha = [\langle \underline{T}, id \rangle, \langle \underline{F}, id \rangle]$$

$$\alpha^{\circ} = \cdots ?$$

Tentemos provar...

$p? \cdot f$

Tentemos provar...

$$
\begin{array}{ll}
 & p? \cdot f \\
= & \quad \{ \; p? = \alpha^{\circ} \cdot \langle p, id \rangle \; \} \\
 & \alpha^{\circ} \cdot \langle p, id \rangle \cdot f
\end{array}
$$

Tentemos provar...

$$
\begin{array}{ll}
 & p? \cdot f \\
= & \{\ p? = \alpha^\circ \cdot \langle p, id \rangle\ \} \\
 & \alpha^\circ \cdot \langle p, id \rangle \cdot f \\
= & \{\ \text{fusão-}\times\ \} \\
 & \alpha^\circ \cdot \langle p \cdot f, id \cdot f \rangle
\end{array}
$$

Tentemos provar...

$$
\begin{aligned}
& p? \cdot f \\
= \quad & \{ \ p? = \alpha^{\circ} \cdot \langle p, id \rangle \ \} \\
& \alpha^{\circ} \cdot \langle p, id \rangle \cdot f \\
= \quad & \{ \ \text{fusão-}\times \ \} \\
& \alpha^{\circ} \cdot \langle p \cdot f, id \cdot f \rangle \\
= \quad & \{ \ \text{natural-}id \text{ duas vezes} \ \} \\
& \alpha^{\circ} \cdot \langle id \cdot p \cdot f, f \cdot id \rangle
\end{aligned}
$$

Tentemos provar...

$$p? \cdot f$$

$$= \qquad \{ \; p? = \alpha^{\circ} \cdot \langle p, id \rangle \; \}$$

$$\alpha^{\circ} \cdot \langle p, id \rangle \cdot f$$

$$= \qquad \{ \; \text{fusão-}\times \; \}$$

$$\alpha^{\circ} \cdot \langle p \cdot f , id \cdot f \rangle$$

$$= \qquad \{ \; \text{natural-}id \text{ duas vezes} \; \}$$

$$\alpha^{\circ} \cdot \langle id \cdot p \cdot f , f \cdot id \rangle$$

$$= \qquad \{ \; \times\text{-absorption} \; \}$$

$$\alpha^{\circ} \cdot (id \times f) \cdot \langle p \cdot f , id \rangle$$

Tentemos provar...

$$
\begin{array}{ll}
 & p? \cdot f \\
= & \quad \{ \ p? = \alpha^{\circ} \cdot \langle p, id \rangle \ \} \\
 & \alpha^{\circ} \cdot \langle p, id \rangle \cdot f \\
= & \quad \{ \ \text{fusão-}\times \ \} \\
 & \alpha^{\circ} \cdot \langle p \cdot f, id \cdot f \rangle \\
= & \quad \{ \ \text{natural-}id \text{ duas vezes} \ \} \\
 & \alpha^{\circ} \cdot \langle id \cdot p \cdot f, f \cdot id \rangle
\end{array}
$$

$$
\begin{array}{ll}
= & \quad \{ \ \times\text{-absorption} \ \} \\
 & \alpha^{\circ} \cdot (id \times f) \cdot \langle p \cdot f, id \rangle \\
= & \quad \{ \ ?? \ \} \\
 & ?? \\
= & \quad \{ \ ?? \ \} \\
 & ??
\end{array}
$$

Relembrar

$$\begin{tikzcd}
2 \times A \arrow[rr, bend left, "\alpha^{\circ}"] \arrow[r, phantom, ""] & \cong & A + A \arrow[ll, bend left, "\alpha"]
\end{tikzcd}$$

$$\alpha = [\langle \underline{T}, id \rangle, \langle \underline{F}, id \rangle]$$

$$\alpha^{\circ} = \cdots ?$$

Grátis de α°

$$\begin{tikzcd}
A + A \arrow[r, leftarrow, "\alpha^\circ"] & 2 \times A
\end{tikzcd}$$

Grátis de α°

$$\begin{tikzcd}
A + A & & 2 \times A \arrow[ll, "\alpha^\circ"'] \\
\\
A' + A' & & 2 \times A' \arrow[ll, "\alpha^\circ"]
\end{tikzcd}$$

Grátis de α°

$$\begin{tikzcd}
A + A \arrow[d, swap, "{''A + A''}"] & & 2 \times A \arrow[ll, swap, "\alpha^\circ"] \arrow[d, "{''2 \times A''}"] \\
A' + A' & & 2 \times A' \arrow[ll, "\alpha^\circ"]
\end{tikzcd}$$

Grátis de α°

$$\begin{tikzcd}
A + A \arrow[d, "f + f"'] & 2 \times A \arrow[l, "\alpha^\circ"'] \arrow[d, "id \times f"] \\
A' + A' & 2 \times A' \arrow[l, "\alpha^\circ"]
\end{tikzcd}$$

$$(f + f) \cdot \alpha^\circ = \alpha^\circ \cdot (id \times f)$$

… e acabemos a prova!

$$
\begin{array}{ll}
 & p? \cdot f \\
= & \{ \ p? = \alpha^\circ \cdot \langle p, id \rangle \ \} \\
 & \alpha^\circ \cdot \langle p, id \rangle \cdot f \\
= & \{ \ \text{fusão-}\times \ \} \\
 & \alpha^\circ \cdot \langle p \cdot f, id \cdot f \rangle \\
= & \{ \ \text{natural-}id \text{ duas vezes} \ \} \\
 & \alpha^\circ \cdot \langle id \cdot p \cdot f, f \cdot id \rangle
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \ \times\text{-absorption} \ \} \\
 & \alpha^\circ \cdot (id \times f) \cdot \langle p \cdot f, id \rangle
\end{array}
$$

… e acabemos a prova!

$$\begin{array}{ll}
 & p? \cdot f \\
= & \quad \{ \; p? = \alpha^{\circ} \cdot \langle p, \mathit{id} \rangle \; \} \\
 & \alpha^{\circ} \cdot \langle p, \mathit{id} \rangle \cdot f \\
= & \quad \{ \; \text{fusão-}\times \; \} \\
 & \alpha^{\circ} \cdot \langle p \cdot f, \mathit{id} \cdot f \rangle \\
= & \quad \{ \; \text{natural-}\mathit{id} \text{ duas vezes} \; \} \\
 & \alpha^{\circ} \cdot \langle \mathit{id} \cdot p \cdot f, f \cdot \mathit{id} \rangle
\end{array}$$

$$\begin{array}{ll}
= & \quad \{ \; \times\text{-absorption} \; \} \\
 & \alpha^{\circ} \cdot (\mathit{id} \times f) \cdot \langle p \cdot f, \mathit{id} \rangle \\
= & \quad \{ \; \text{grátis de } \alpha^{\circ} \; \} \\
 & (f + f) \cdot \alpha^{\circ} \cdot \langle p \cdot f, \mathit{id} \rangle
\end{array}$$

... e acabemos a prova!

$$
\begin{aligned}
& p? \cdot f \\
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$$
\begin{aligned}
= & \quad \{ \ \times\text{-absorption} \ \} \\
& \alpha^{\circ} \cdot (id \times f) \cdot \langle p \cdot f, id \rangle \\
= & \quad \{ \ \text{grátis de } \alpha^{\circ} \ \} \\
& (f + f) \cdot \alpha^{\circ} \cdot \langle p \cdot f, id \rangle \\
= & \quad \{ \ p? = \alpha^{\circ} \cdot \langle p, id \rangle \ \} \\
& (f + f) \cdot (p \cdot f)?
\end{aligned}
$$

🙂

2ª parte da matéria

Como nasce um programa?

O que é um programa

Como nasce um programa?

Como nasceram os algoritmos célebres?

Como nasce um programa?

Como nasceram os algoritmos célebres?

Por exemplo, “**quicksort**”?

Como nasce um programa?

O que é um programa

Como nasce um programa?

$$\begin{cases} a \times 0 = 0 \\ a \times 1 = a \\ a \times (b + c) = a \times b + a \times c \end{cases}$$

Como nasce um programa?

$$\begin{cases} a \times 0 = 0 \\ a \times 1 = a \\ a \times (b + 1) = a \times b + a \times 1 \end{cases}$$

Como nasce um programa?

$$\begin{cases} a \times 0 = 0 \\ a \times 1 = a \\ a \times (b + 1) = a \times b + a \end{cases}$$

Como nasce um programa?

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Como nasce um programa?

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Como nasce um programa?

$$\begin{cases} a \times 0 = 0 \\ a \times (b + 1) = a \times b + a \end{cases}$$

O que é um programa?

$$\begin{cases} a \times 0 = 0 \\ a \times (b + 1) = a \times b + a \end{cases}$$

O que **é** um programa?

$$\begin{cases} a \times 0 = 0 \\ \\ a \times (b + 1) = a + a \times b \end{cases}$$

O que **é** um programa?

$$\begin{cases} (a\times)\ 0 = 0 \\ \\ (a\times)\ (b+1) = a + (a\times)\ b \end{cases}$$

O que é um programa?

$$\begin{cases} (a\times)\ 0 = 0 \\ \\ (a\times)\ (b+1) = (a+)\ ((a\times)\ b) \end{cases}$$

O que **é** um programa?

$$\begin{cases} (a\times)\ (\underline{0}\ x) = \underline{0}\ x \\ \\ (a\times)\ (b+1) = (a+)\ ((a\times)\ b) \end{cases}$$

O que **é** um programa?

$$\begin{cases} (a\times)\ (\underline{0}\ x) = \underline{0}\ x \\ \\ (a\times)\ (\mathit{succ}\ b) = (a+)\ ((a\times)\ b) \end{cases}$$

O que **é** um programa?

$$\begin{cases} ((a\times)\cdot\underline{0})\ x = \underline{0}\ x \\ \\ (a\times)\ (succ\ b) = (a+)\ ((a\times)\ b) \end{cases}$$

O que é um programa?

$$\begin{cases} ((a\times)\cdot\underline{0})\ x = \underline{0}\ x \\ \\ ((a\times)\cdot succ)\ b = (a+)\ ((a\times)\ b) \end{cases}$$

O que **é** um programa?

$$\begin{cases} ((a\times)\cdot\underline{0})\ x = \underline{0}\ x \\ \\ ((a\times)\cdot succ)\ b = ((a+)\cdot(a\times))\ b \end{cases}$$

O que é um programa?

$$\begin{cases} (a\times)\cdot\underline{0} = \underline{0} \\ \\ ((a\times)\cdot succ)\ b = ((a+)\cdot(a\times))\ b \end{cases}$$

O que é um programa?

$$\begin{cases} (a\times)\cdot\underline{0} = \underline{0} \\ \\ (a\times)\cdot succ = (a+)\cdot(a\times) \end{cases}$$

O que é um programa?

$$\begin{cases} (a\times)\cdot\underline{0} = \underline{0} \\ (a\times)\cdot succ = (a+)\cdot(a\times) \end{cases}$$

O que é um programa?

$$[(a\times)\cdot\underline{0},(a\times)\cdot succ]=[\underline{0},(a+)\cdot(a\times)]$$

O que é um programa?

$$(a\times)\cdot[\underline{0}, succ] = [\underline{0}, (a+)\cdot(a\times)]$$

O que é um programa?

$$(a\times)\cdot[\underline{0}, succ] = [\underline{0}\cdot id, (a+)\cdot(a\times)]$$

O que é um programa?

$$(a\times)\cdot[\underline{0}, succ] = [\underline{0}, (a+)]\cdot(id + (a\times))$$

$$(a\times)\cdot[\underline{0}, succ] = [\underline{0}, (a+)]\cdot(id + (a\times))$$

$$\begin{tikzcd}
? \arrow[d, "(a\times)"'] \\
?
\end{tikzcd}$$

$$(a\times)\cdot[\underline{0}, succ] = [\underline{0}, (a+)]\cdot(id + (a\times))$$

$$\begin{tikzcd}
? \arrow[d, "(a\times)"'] & & ? \arrow[ll, "{[\underline{0}, succ]}"'] \\
?
\end{tikzcd}$$

$$(a\times)\cdot[\underline{0}, succ] = [\underline{0}, (a+)]\cdot(id + (a\times))$$

$$\begin{tikzcd}
? \arrow[d, "(a\times)"'] & ? \arrow[l, "{[\underline{0}, succ]}"'] \arrow[d, "id + (a\times)"] \\
? & ?
\end{tikzcd}$$

$$(a\times)\cdot[\underline{0}, succ] = [\underline{0}, (a+)]\cdot(id + (a\times))$$

$$\begin{tikzcd}
? \arrow[d, "(a\times)"'] & ? \arrow[l, "{[\underline{0}, succ]}"'] \arrow[d, "id + (a\times)"] \\
? & ? \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Um programa é um “rectângulo”

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(a\times)"'] & ? \arrow[l, "{[\underline{0}, succ]}"'] \arrow[d, "id + (a\times)"] \\
\mathbb{N}_0 & ? \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Um programa é um “rectângulo”

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(a\times)"'] & ? + ? \arrow[l, "{[\underline{0}, succ]}"'] \arrow[d, "id + (a\times)"] \\
\mathbb{N}_0 & ? + ? \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Um programa é um "rectângulo"

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(a\times)"'] & ? + \mathbb{N}_0 \arrow[l, "{[\underline{0}, succ]}"'] \arrow[d, "id + (a\times)"] \\
\mathbb{N}_0 & ? + \mathbb{N}_0 \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Um programa é um “rectângulo”

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(a\times)"'] & A + \mathbb{N}_0 \arrow[l, "{[\underline{0}, succ]}"'] \arrow[d, "id + (a\times)"] \\
\mathbb{N}_0 & A + \mathbb{N}_0 \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Existirá $[\underline{0}, succ]^\circ$?

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, "{[\underline{0}, succ]^\circ}"] \arrow[d, "(a\times)"'] & A + \mathbb{N}_0 \arrow[d, "id + (a\times)"] \\
\mathbb{N}_0 & A + \mathbb{N}_0 \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Existirá $[\underline{0}, \mathit{succ}]^{\circ}$?

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(a \times)"'] & A + \mathbb{N}_0 \arrow[l, "{[\underline{0}, \mathit{succ}]}"'] \arrow[d, "id + (a \times)"] \\
\mathbb{N}_0 & A + \mathbb{N}_0 \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Existirá $[\underline{0}, succ]^\circ$?

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, "{[\underline{0}, succ]^\circ}"] \arrow[d, "(a\times)"'] & A + \mathbb{N}_0 \arrow[d, "id + (a\times)"] \\
\mathbb{N}_0 & A + \mathbb{N}_0 \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

Existirá $[\underline{0}, succ]^\circ$?

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, "{[\underline{0}, succ]^\circ}"] \arrow[d, "(a\times)"'] & A + \mathbb{N}_0 \arrow[d, "id + (a\times)"] \\
\mathbb{N}_0 & A + \mathbb{N}_0 \arrow[l, "{[\underline{0}, (a+)]}"]
\end{tikzcd}$$

$$(a\times) = [\underline{0}, (a+)] \cdot (id + (a\times)) \cdot [\underline{0}, succ]^\circ (?)$$

Existirá $[\underline{0}, succ]^\circ$?

Chamemos **out** : $\mathbb{N}_0 \rightarrow A + \mathbb{N}_0$ a essa (possível) inversa:

$$\mathbf{out} \cdot [\underline{0}, succ] = id$$

Existirá $[\underline{0}, succ]^\circ$?

Chamemos **out** : $\mathbb{N}_0 \rightarrow A + \mathbb{N}_0$ a essa (possível) inversa:

$$
\begin{array}{ll}
 & \mathbf{out} \cdot [\underline{0}, succ] = id \\
\Leftrightarrow & \{ \text{ fusão-}+ \ \} \\
 & [\mathbf{out} \cdot \underline{0}, \mathbf{out} \cdot succ] = id
\end{array}
$$

Existirá $[\underline{0}, succ]^\circ$?

Chamemos **out** : $\mathbb{N}_0 \rightarrow A + \mathbb{N}_0$ a essa (possível) inversa:

$$
\begin{aligned}
& \mathbf{out} \cdot [\underline{0}, succ] = id \\
\Leftrightarrow \quad & \{ \text{ fusão-}+ \} \\
& [\mathbf{out} \cdot \underline{0}, \mathbf{out} \cdot succ] = id \\
\Leftrightarrow \quad & \{ \text{ universal-}+ \} \\
& \left\{ \begin{array}{l} \mathbf{out} \cdot \underline{0} = i_1 \\ \mathbf{out} \cdot succ = i_2 \end{array} \right.
\end{aligned}
$$

Existirá $[\underline{0}, \mathit{succ}]^{\circ}$?

$$\begin{cases} \mathbf{out} \cdot \underline{0} = i_1 \\ \mathbf{out} \cdot \mathit{succ} = i_2 \end{cases}$$

Existirá $[\underline{0}, succ]^{\circ}$?

$$\left\{\begin{array}{l}\mathbf{out} \cdot \underline{0} = i_1 \\ \mathbf{out} \cdot succ = i_2\end{array}\right.$$

$$\Leftrightarrow \qquad \{ \text{ introduzir variáveis } \}$$

$$\left\{\begin{array}{l}\mathbf{out}\ (\underline{0}\ x) = i_1\ x \\ \mathbf{out}\ (succ\ x) = i_2\ x\end{array}\right.$$

Existirá $[\underline{0}, succ]^{\circ}$?

$$\begin{cases} \mathbf{out} \cdot \underline{0} = i_1 \\ \mathbf{out} \cdot succ = i_2 \end{cases}$$

$$\Leftrightarrow \qquad \{ \text{ introduzir variáveis } \}$$

$$\begin{cases} \mathbf{out}\ (\underline{0}\ x) = i_1\ x \\ \mathbf{out}\ (succ\ x) = i_2\ x \end{cases}$$

$$\Leftrightarrow \qquad \{ \text{ simplificar } \}$$

$$\begin{cases} \mathbf{out}\ 0 = i_1\ x \\ \mathbf{out}\ (x + 1) = i_2\ x \end{cases}$$

Existirá $[\underline{0}, succ]^{\circ}$?

$$\left\{\begin{array}{l}\mathbf{out} \cdot \underline{0} = i_1 \\ \mathbf{out} \cdot succ = i_2\end{array}\right.$$

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$$\left\{\begin{array}{l}\mathbf{out}\ (\underline{0}\ x) = i_1\ x \\ \mathbf{out}\ (succ\ x) = i_2\ x\end{array}\right.$$

$$\Leftrightarrow \qquad \{ \text{ simplificar } \}$$

$$\left\{\begin{array}{l}\mathbf{out}\ 0 = i_1\ x \\ \mathbf{out}\ (x+1) = i_2\ x\end{array}\right.$$

$$\left\{\begin{array}{l}\mathbf{out}\ 0 = i_1\ x \\ \mathbf{out}\ (x+1) = i_2\ x\end{array}\right.$$

?

Problema

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "{\begin{cases} \mathbf{out}\ 0 = i_1\ x \\ \mathbf{out}\ (x+1) = i_2\ x \end{cases}}"] \arrow[r, phantom, "?"] & A + \mathbb{N}_0 \arrow[l, bend left, "{\mathbf{in} = [\underline{0}, succ]}"]
\end{tikzcd}$$

?

Solução

$$\begin{cases} \mathbf{out}\ 0 = i_1\ () \\ \mathbf{out}\ (x+1) = i_2\ x \end{cases}$$

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \mathbf{1} + \mathbb{N}_0 \arrow[l, bend left, "\mathbf{in} = {[\underline{0}, \mathit{succ}]}"]
\end{tikzcd}$$

Solução

$$\begin{cases} \mathbf{out}\ 0 = i_1\ () \\ \mathbf{out}\ (x+1) = i_2\ x \end{cases}$$

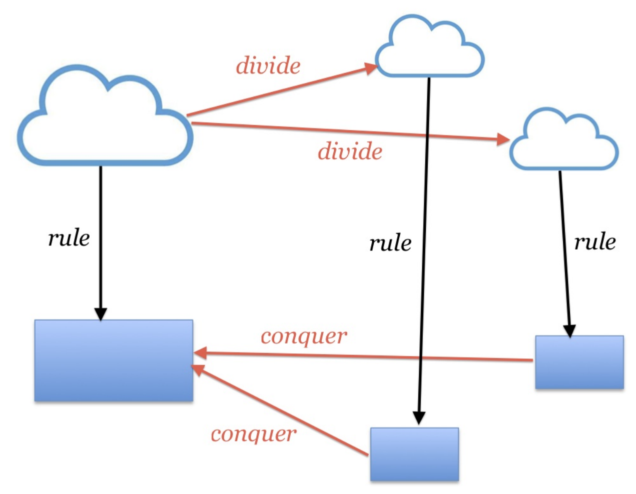
$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left] \arrow[r, phantom, "\cong"] \arrow[d, "(a\times)"'] & 1+\mathbb{N}_0 \arrow[l, bend left, "{\mathbf{in}=[\underline{0},succ]}"] \arrow[d, "id+(a\times)"] \\
\mathbb{N}_0 & 1+\mathbb{N}_0 \arrow[l, bend left, "{[\underline{0},(a+)]}"]
\end{tikzcd}$$

Diagrama

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[rr, bend left, "\mathsf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "k"'] & & 1 + \mathbb{N}_0 \arrow[ll, bend left, "\mathsf{in}"] \arrow[d, "id+k"] \\
\mathbb{N}_0 & & 1 + \mathbb{N}_0 \arrow[ll, bend left, "{[\underline{0}, a+]}"]
\end{tikzcd}$$

$$(a\times) = k \textbf{ where } k = [\underline{0}, (a+)] \cdot (id + k) \cdot \mathsf{out}$$

Antecipação...



Variações no diagrama

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1+\mathbb{N}_0 \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id+k"] \\
\mathbb{N}_0 & 1+\mathbb{N}_0 \arrow[l, bend left, "{[\underline{1},(a\times)]}"]
\end{tikzcd}$$

Variações no diagrama

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "id+k"] \\
\mathbb{N}_0 & 1 + \mathbb{N}_0 \arrow[l, bend left, "{[\underline{1}, (a\times)]}"]
\end{tikzcd}$$

$$a^b = k\ b \ \textbf{where}\ k = [\underline{1}, (a\times)] \cdot (id + k) \cdot \mathbf{out}$$

Generalização do diagrama

$$\begin{cases} a^{0} = 1 \\ a^{b+1} = a \times a^{b} \end{cases}$$

Generalização do diagrama

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[rr, bend left, "\mathbf{out}"] \arrow[d, "k\ g"'] & \cong & 1+\mathbb{N}_0 \arrow[ll, bend left, "\mathbf{in}"] \arrow[d, "id+k\ g"] \\
A & & 1+A \arrow[ll, bend left, "g"]
\end{tikzcd}$$

$$k\ g = g \cdot (id + k\ g) \cdot \mathbf{out}$$

Generalização do diagrama

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "\llparenthesis g \rrparenthesis"'] & 1+\mathbb{N}_0 \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "id+\llparenthesis g \rrparenthesis"] \\
A & 1+A \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\llparenthesis g \rrparenthesis = g \cdot (id + \llparenthesis g \rrparenthesis) \cdot \mathbf{out}$$

Definição

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, "\mathsf{out}"] \arrow[d, "\llparenthesis\, g \,\rrparenthesis"'] & 1 + \mathbb{N}_0 \arrow[d, "id + \llparenthesis\, g \,\rrparenthesis"] \\
A & 1 + A \arrow[l, "g"]
\end{tikzcd}$$

$$\llparenthesis\, g \,\rrparenthesis = g \cdot (id + \llparenthesis\, g \,\rrparenthesis) \cdot \mathsf{out}$$

Propriedade

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(\!| g |\!)"'] & 1 + \mathbb{N}_0 \arrow[l, "\mathsf{in}"'] \arrow[d, "id + (\!| g |\!)"] \\
A & 1 + A \arrow[l, "g"]
\end{tikzcd}$$

$$(\!| g |\!) \cdot \mathsf{in} = g \cdot (id + (\!| g |\!))$$

Propriedade universal

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "k"'] & 1 + \mathbb{N}_0 \arrow[l, "\mathbf{in}"'] \arrow[d, "id+k"] \\
A & 1 + A \arrow[l, "g"]
\end{tikzcd}$$

$$k = (\!| g |\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot (id + k)$$

Universal-$\times$:

$$k = \langle f, g \rangle \;\Leftrightarrow\; \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

Universal-$\times$:

$$k = \langle f, g \rangle \iff \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$+$:

$$k = [f, g] \iff \begin{cases} k \cdot i_1 = f \\ k \cdot i_2 = g \end{cases}$$

Universal-$\times$:

$$k = \langle f, g \rangle \;\Leftrightarrow\; \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$+$:

$$k = [f, g] \;\Leftrightarrow\; \begin{cases} k \cdot i_1 = f \\ k \cdot i_2 = g \end{cases}$$

Universal-expo:

$$k = \overline{f} \;\Leftrightarrow\; ap \cdot (k \times id) = f$$

Universal-$\times$:

$$k = \langle f , g \rangle \;\Leftrightarrow\; \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

Universal-$+$:

$$k = [f , g] \;\Leftrightarrow\; \left\{ \begin{array}{l} k \cdot i_1 = f \\ k \cdot i_2 = g \end{array} \right.$$

Universal-expo:

$$k = \overline{f} \;\Leftrightarrow\; ap \cdot (k \times id) = f$$

Universal-cata:

$$k = (\!| g |\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot (id + k)$$

Catamorfismo

$$\underbrace{\textit{cata}}_{\textit{para baixo}}\ (\kappa\alpha\tau\alpha)$$

Catamorfismo

$$\underbrace{\textit{cata}}_{\textit{para baixo}} \; (\kappa\alpha\tau\alpha) + \underbrace{\textit{morfismo}}_{\text{transformação}} \qquad (\mu o\rho\phi\iota\sigma\mu o\zeta)$$

“Transformação descendente”

Propriedade universal-cata

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "k"'] & 1 + \mathbb{N}_0 \arrow[l, "\mathbf{in}"'] \arrow[d, "id+k"] \\
A & 1 + A \arrow[l, "g"]
\end{tikzcd}$$

$$k = (\!| g |\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot (id + k)$$

Cancelamento-cata

$$k = (\!| g |\!) \Leftrightarrow k \cdot \mathbf{in} = g \cdot (id + k)$$

Cancelamento-cata

$$(\!|\, g \,|\!) = (\!|\, g \,|\!) \Leftrightarrow (\!|\, g \,|\!) \cdot \textbf{in} = g \cdot (id + (\!|\, g \,|\!))$$

Cancelamento-cata

$$(\!| g |\!) \cdot \mathbf{in} = g \cdot (id + (\!| g |\!))$$

Definição-cata

$$(\!|\, g \,|\!) = g \cdot (id + (\!|\, g \,|\!)) \cdot \mathbf{out}$$

Definição-cata

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, "\mathbf{out}"] \arrow[d, "(\!| g |\!)"'] & 1 + \mathbb{N}_0 \arrow[d, "id + (\!| g |\!)"] \\
A & 1 + A \arrow[l, "g"]
\end{tikzcd}$$

$$(\!| g |\!) = g \cdot (id + (\!| g |\!)) \cdot \mathbf{out}$$

Propriedade universal-cata

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "k"'] & 1 + \mathbb{N}_0 \arrow[l, "\mathbf{in} = [\underline{0}, succ]"'] \arrow[d, "id + k"] \\
A & 1 + A \arrow[l, "g"]
\end{tikzcd}$$

$$k = \llparenthesis\, g \,\rrparenthesis \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot (id + k)$$

Reflexão-cata

$$k = (\!| g |\!) \Leftrightarrow k \cdot \mathbf{in} = g \cdot (id + k)$$

Reflexão-cata

$$id = (\!| g |\!) \Leftrightarrow id \cdot \mathbf{in} = g \cdot (id + id)$$

Reflexão-cata

$$id = (\!|\, g \,|\!) \Leftrightarrow \mathbf{in} = g$$

Reflexão-cata

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "id = (\!| \mathbf{in} |\!)"'] & 1 + \mathbb{N}_0 \arrow[l, "\mathbf{in} = [\underline{0}, succ]"'] \arrow[d, "id + (\!| \mathbf{in} |\!)"] \\
\mathbb{N}_0 & 1 + \mathbb{N}_0 \arrow[l, "\mathbf{in}"]
\end{tikzcd}$$

$$(\!| \mathbf{in} |\!) = id$$

Propriedade fusão-cata

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "\llparenthesis g \rrparenthesis"'] & 1+\mathbb{N}_0 \arrow[l, "\mathbf{in}=[\underline{0}, succ]"'] \arrow[d, "id+\llparenthesis g \rrparenthesis"] \\
A \arrow[d, "f"'] & 1+A \arrow[l, "g"] \arrow[d, "id+f"] \\
B & 1+B \arrow[l, "h"]
\end{tikzcd}$$

Propriedade fusão-cata

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(\!|\, g \,|\!)"'] \arrow[dd, bend right=60, "(\!|\, h \,|\!)"'] & & 1+\mathbb{N}_0 \arrow[ll, "\mathbf{in}={[\underline{0}, succ]}"'] \arrow[d, "id+(\!|\, g \,|\!)"] \\
A \arrow[d, "f"'] & & 1+A \arrow[ll, "g"] \arrow[d, "id+f"] \\
B & & 1+B \arrow[ll, "h"]
\end{tikzcd}$$

Fusão-cata

$$f \cdot (\!|\, g \,|\!) = (\!|\, h \,|\!)$$

Fusão-cata

$$
\begin{array}{ll}
 & f \cdot (\!| g |\!) = (\!| h |\!) \\
\Leftrightarrow & \qquad \{ \text{ universal-cata para } k = f \cdot (\!| g |\!) \ \} \\
 & f \cdot (\!| g |\!) \cdot \mathbf{in} = h \cdot (id + f \cdot (\!| g |\!))
\end{array}
$$

Fusão-cata

$$
\begin{array}{ll}
 & f \cdot (\!| g |\!) = (\!| h |\!) \\
\Leftrightarrow & \quad \{ \text{ universal-cata para } k = f \cdot (\!| g |\!) \ \} \\
 & f \cdot (\!| g |\!) \cdot \mathbf{in} = h \cdot (id + f \cdot (\!| g |\!)) \\
\Leftrightarrow & \quad \{ \text{ cancelamento-cata } \} \\
 & f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id + f \cdot (\!| g |\!))
\end{array}
$$

Fusão-cata

$$f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id \cdot id + f \cdot (\!| g |\!))$$

Fusão-cata

$$
\begin{aligned}
& f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id \cdot id + f \cdot (\!| g |\!)) \\
\Leftrightarrow \quad & \qquad \{ \text{ functor-}+ \text{ } \} \\
& f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id + f) \cdot (id + (\!| g |\!))
\end{aligned}
$$

Fusão-cata

$$f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id \cdot id + f \cdot (\!| g |\!))$$

$$\Leftrightarrow \qquad \{ \text{ functor-}+ \}$$

$$f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id + f) \cdot (id + (\!| g |\!))$$

$$\Leftarrow \qquad \{ \text{ Leibniz } \}$$

$$f \cdot g = h \cdot (id + f)$$

Fusão-cata

$$
\begin{array}{ll}
 & f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id \cdot id + f \cdot (\!| g |\!)) \\
\Leftrightarrow & \quad \{ \text{ functor-}+ \ \} \\
 & f \cdot g \cdot (id + (\!| g |\!)) = h \cdot (id + f) \cdot (id + (\!| g |\!)) \\
\Leftarrow & \quad \{ \text{ Leibniz } \} \\
 & f \cdot g = h \cdot (id + f)
\end{array}
$$

Em suma:

$$f \cdot (\!| g |\!) = (\!| h |\!) \quad \Leftarrow \quad f \cdot g = h \cdot (id + f)$$

Cálculo
de
Programas

Aula T06 - online

O que **é** um programa?

Por exemplo, “**quicksort**”?

O que é um programa?

Por exemplo, “**quicksort**”?

Programas são “rectângulos” 🙂

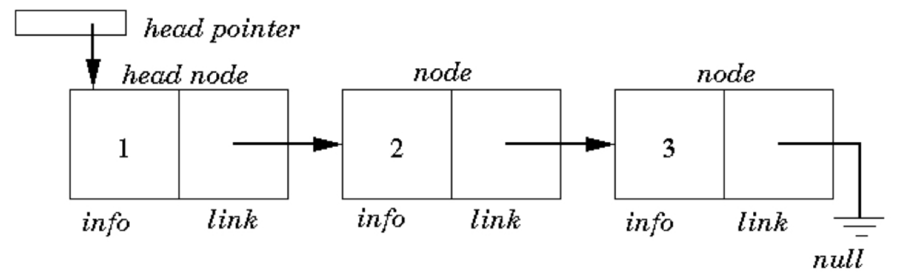
O que é um programa?

Por exemplo, “**quicksort**”?

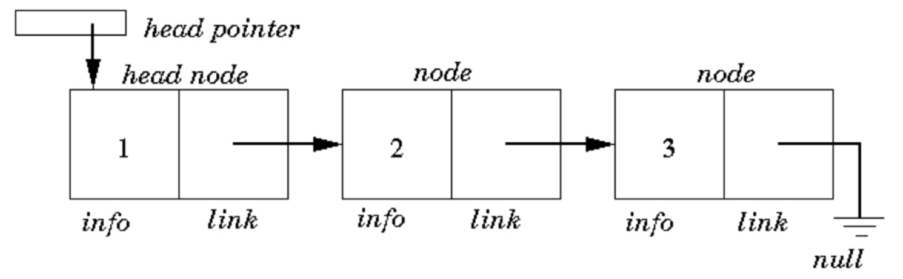
Programas são “rectângulos” 🙂

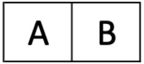
Mas há ainda um caminho a percorrer...

Lista ligada



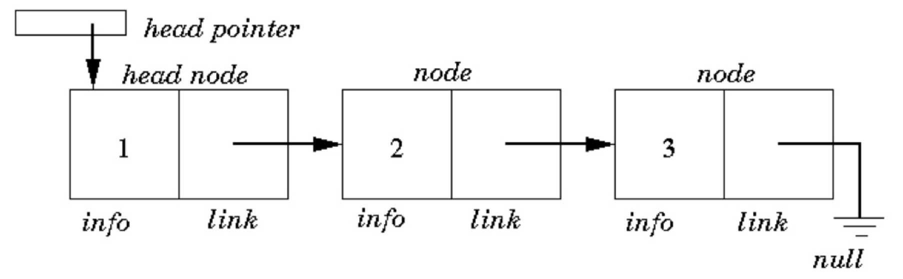
Lista ligada

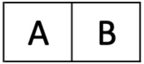




$A \times B$

Lista ligada

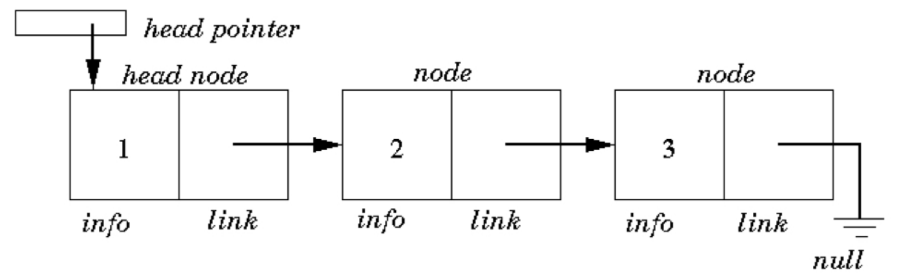




$A \times B$

$1 + A$

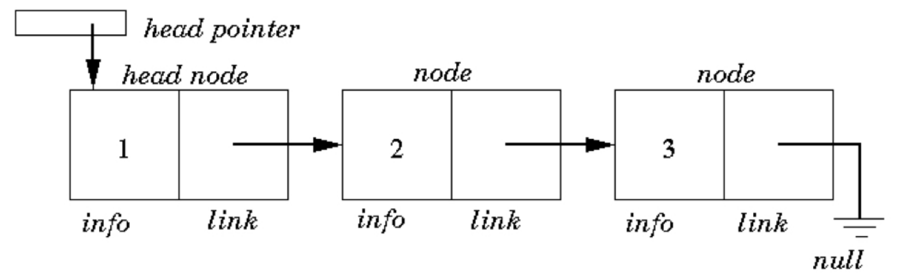
Lista ligada



"pointer"

$L = 1 + N$

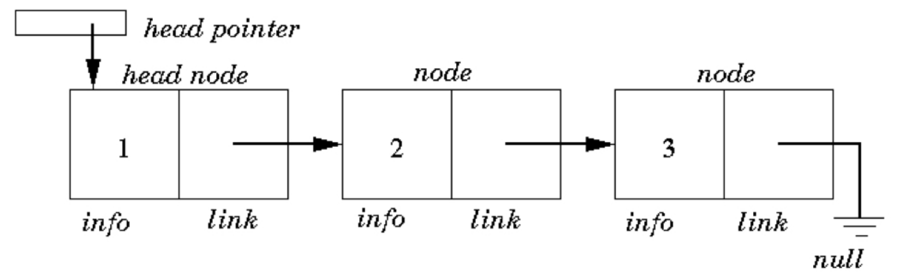
Lista ligada



“pointer” $L = 1 + N$

“node” $N = \mathbb{N}_0 \times L$

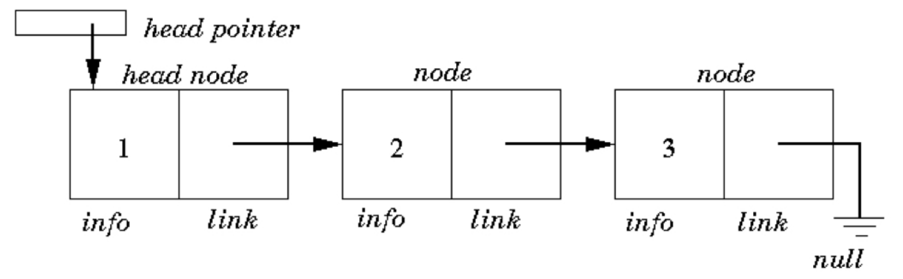
Lista ligada



Substitutuição

$$L = 1 + \mathbb{N}_0 \times L$$

Lista ligada



De facto

$$L \cong 1 + \mathbb{N}_0 \times L$$

Lista ligada

$$L \cong 1 + \mathbb{N}_0 \times L$$

Lista ligada

$$L \cong 1 + \mathbb{N}_0 \times L$$

$$\begin{tikzcd} L \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathsf{in}"] \end{tikzcd}$$

Lista ligada

$$L \cong 1 + \mathbb{N}_0 \times L$$

$$\begin{tikzcd} L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"] \end{tikzcd}$$

$$\mathbf{in} = [\mathit{nil}, \mathit{cons}]$$

Lista ligada

$$L \cong 1 + \mathbb{N}_0 \times L$$

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"]
\end{tikzcd}$$

$$\mathbf{in} = [nil, cons] \qquad \begin{cases} nil\ \_ = [\,] \\ cons\ (a, l) = a : l \end{cases}$$

Lista ligada

$$L \;\cong\; 1 + \mathbb{N}_0 \times L$$

$$\begin{tikzcd} L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"] \end{tikzcd}$$

$$\mathbf{in} = [nil\,, cons]$$

$$\left\{ \begin{array}{l} nil\ \_ = [\,] \\ cons\ (a, l) = a : l \end{array} \right.$$

$$\mathbf{out} \cdot \mathbf{in} = id$$

Lista ligada

$$\begin{tikzcd} L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"] \end{tikzcd}$$

$$\left\{ \begin{array}{l} \mathbf{out} \cdot \mathbf{in} = id \\ \mathbf{in} = [nil, cons] \end{array} \right.$$

Lista ligada

$$\begin{tikzcd} L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"] \end{tikzcd}$$

$$\left\{ \begin{array}{l} \mathbf{out} \cdot \mathbf{in} = id \\ \mathbf{in} = [nil, cons] \end{array} \right. \Rightarrow$$

Lista ligada

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"]
\end{tikzcd}$$

$$\left\{\begin{array}{l} \mathbf{out} \cdot \mathbf{in} = id \\ \mathbf{in} = [nil, cons] \end{array}\right. \Rightarrow \left\{\begin{array}{l} \mathbf{out}\ [\,] = i_1\ () \\ \mathbf{out}\ (a : x) = i_2\ (a, x) \end{array}\right.$$

Comprimento de uma lista

Recordar a **concatenação** de listas $x +\!\!+\, y$

Comprimento de uma lista

Recordar a **concatenação** de listas $x +\!\!+\, y$ que é tal que

$$[\,] +\!\!+\, x = x$$

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Recordar a **concatenação** de listas $x \mathbin{+\!\!+} y$ que é tal que

$$[\,] \mathbin{+\!\!+} x = x$$
$$[a] \mathbin{+\!\!+} x = a : x$$

Comprimento de uma lista

Recordar a **concatenação** de listas $x +\!\!+\ y$ que é tal que

$$
\begin{aligned}
[\,] +\!\!+\ x &= x \\
[a] +\!\!+\ x &= a : x
\end{aligned}
$$

Propriedades:

Comprimento de uma lista

Recordar a **concatenação** de listas $x \mathbin{+\!\!+} y$ que é tal que

$$
\begin{aligned}
[\,] \mathbin{+\!\!+} x &= x \\
[a] \mathbin{+\!\!+} x &= a : x
\end{aligned}
$$

Propriedades:

$$\mathit{length}\ [\,] = 0$$

Comprimento de uma lista

Recordar a **concatenação** de listas $x \mathbin{+\!\!+} y$ que é tal que

$$[\,] \mathbin{+\!\!+} x = x$$
$$[a] \mathbin{+\!\!+} x = a : x$$

Propriedades:

$$\mathit{length}\ [\,] = 0$$
$$\mathit{length}\ [a] = 1$$

Comprimento de uma lista

Recordar a **concatenação** de listas $x + y$ que é tal que

$$
\begin{aligned}
[\,] + x &= x \\
[a] + x &= a : x
\end{aligned}
$$

Propriedades:

$$
\begin{aligned}
\mathit{length}\ [\,] &= 0 \\
\mathit{length}\ [a] &= 1 \\
\mathit{length}\ (x + y) &= \mathit{length}\ x + \mathit{length}\ y
\end{aligned}
$$

Comprimento de uma lista

$$\begin{aligned}
&\mathit{length}\ [\,] = 0\\
&\mathit{length}\ [a] = 1\\
&\mathit{length}\ ([a] \mathbin{+\!\!+} y) = \mathit{length}\ [a] + \mathit{length}\ y
\end{aligned}$$

Comprimento de uma lista

$$length\ [\,] = 0$$
$$length\ [a] = 1$$
$$length\ ([a] \mathbin{+\!\!+} y) = length\ [a] + length\ y$$

$$length\ (a : y) = 1 + length\ y$$

Length of a list

$$\begin{aligned} \mathit{length}\ [\,] &= 0 \\ \mathit{length}\ (a : y) &= 1 + \mathit{length}\ y \end{aligned}$$

Length of a list

$$\mathit{length}\ [\,] = 0$$
$$\mathit{length}\ (a : y) = 1 + \mathit{length}\ y$$

$$(\mathit{length} \cdot \mathit{nil})\ \_ = \underline{0}\ \_$$
$$(\mathit{length} \cdot \mathit{cons})\ (a, y) = 1 + \mathit{length}\ (\pi_2\ (a, y))$$

Length of a list

$$\begin{aligned} \mathit{length}\ [\,] &= 0 \\ \mathit{length}\ (a : y) &= 1 + \mathit{length}\ y \end{aligned}$$

$$\begin{aligned} (\mathit{length} \cdot \mathit{nil})\ \_ &= \underline{0}\ \_ \\ (\mathit{length} \cdot \mathit{cons})\ (a, y) &= 1 + \mathit{length}\ (\pi_2\ (a, y)) \end{aligned}$$

$$\begin{aligned} \mathit{length} \cdot \mathit{nil} &= \underline{0} \\ \mathit{length} \cdot \mathit{cons} &= \mathit{succ} \cdot \mathit{length} \cdot \pi_2 \end{aligned}$$

Length of a list

$$\mathit{length} \cdot \mathit{nil} = \underline{0}$$

$$\mathit{length} \cdot \mathit{cons} = \mathit{succ} \cdot \mathit{length} \cdot \pi_2$$

Length of a list

$$length \cdot nil = \underline{0}$$
$$length \cdot cons = succ \cdot length \cdot \pi_2$$

$$length \cdot [nil, cons] = [\underline{0}, succ \cdot \pi_2] \cdot (id + id \times length)$$

Length of a list

$$
\begin{aligned}
\mathit{length} \cdot \mathit{nil} &= \underline{0} \\
\mathit{length} \cdot \mathit{cons} &= \mathit{succ} \cdot \mathit{length} \cdot \pi_2
\end{aligned}
$$

$$\mathit{length} \cdot [\mathit{nil}, \mathit{cons}] = [\underline{0}, \mathit{succ} \cdot \pi_2] \cdot (\mathit{id} + \mathit{id} \times \mathit{length})$$

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "\mathit{length}"'] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "\mathit{id}+\mathit{id}\times\mathit{length}"] \\
\mathbb{N}_0 & 1 + \mathbb{N}_0 \times \mathbb{N}_0 \arrow[l, bend left, "{[\underline{0}, \mathit{succ}\cdot\pi_2]}"]
\end{tikzcd}$$

$$
\begin{cases}
\mathbf{in} = [\mathit{nil}, \mathit{cons}] \\
\mathbf{out} = \mathbf{in}^{\circ}
\end{cases}
$$

Somatório de uma lista

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "sum"'] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "id+id\times sum"] \\
\mathbb{N}_0 & 1 + \mathbb{N}_0 \times \mathbb{N}_0 \arrow[l, bend left, "{[\underline{0}, add]}"]
\end{tikzcd}$$

$$\begin{cases} \mathbf{in} = [nil\,, cons] \\ \mathbf{out} = \mathbf{in}^\circ \end{cases}$$

$$sum \cdot \mathbf{in} = [\underline{0}, add] \cdot (id + id \times sum)$$

Generalização

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id + id \times k"] \\
B & 1 + \mathbb{N}_0 \times B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Generalização

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id+id\times k"] \\
B & 1 + \mathbb{N}_0 \times B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id+k"] \\
B & 1 + B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Generalização

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id + id \times k"] \\
B & 1 + \mathbb{N}_0 \times B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id + k"] \\
B & 1 + B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$k \cdot \mathbf{in} = g \cdot \underbrace{id + id \times k}_{\mathbf{F}\ k}$$

Generalização

$$\begin{tikzcd}
L \arrow[rr, bend left, "\mathbf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "k"'] & & 1 + \mathbb{N}_0 \times L \arrow[ll, bend left, "\mathbf{in}"] \arrow[d, "id + id \times k"] \\
B & & 1 + \mathbb{N}_0 \times B \arrow[ll, bend left, "g"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[rr, bend left, "\mathbf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "k"'] & & 1 + \mathbb{N}_0 \arrow[ll, bend left, "\mathbf{in}"] \arrow[d, "id + k"] \\
B & & 1 + B \arrow[ll, bend left, "g"]
\end{tikzcd}$$

$$k \cdot \mathbf{in} = g \cdot \underbrace{id + id \times k}_{\mathbf{F}\ k}$$

$$k \cdot \mathbf{in} = g \cdot \underbrace{id + k}_{\mathbf{F}\ k}$$

Abstração ($\mathbb{N}_0$)

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1+\mathbb{N}_0 \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "id+k"] \\
B & 1+B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Abstração ($\mathbb{N}_0$)

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1+\mathbb{N}_0 \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id+k"] \\
B & 1+B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\left\{ \begin{array}{l} \mathbf{F}\ k = id + k \\ \mathbf{F}\ X = 1 + X \end{array} \right.$$

Abstração ($\mathbb{N}_0$)

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "id+k"] \\
B & 1 + B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & \mathbf{F}\ \mathbb{N}_0 \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "\mathbf{F}\ k"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\begin{cases} \mathbf{F}\ k = id + k \\ \mathbf{F}\ X = 1 + X \end{cases}$$

Abstração (listas)

$$\begin{tikzcd}
L \arrow[rr, bend left, "\mathbf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "k"'] & & 1 + \mathbb{N}_0 \times L \arrow[ll, bend left, "\mathbf{in}"] \arrow[d, "id + id \times k"] \\
B & & 1 + \mathbb{N}_0 \times B \arrow[ll, bend left, "g"]
\end{tikzcd}$$

Abstração (listas)

$$\begin{tikzcd}
L \arrow[rr, bend left, "\mathbf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "k"'] & & 1 + \mathbb{N}_0 \times L \arrow[ll, bend left, "\mathbf{in}"] \arrow[d, "id + id \times k"] \\
B & & 1 + \mathbb{N}_0 \times B \arrow[ll, bend left, "g"]
\end{tikzcd}$$

$$\left\{\begin{array}{l} \mathbf{F}\ k = id + id \times k \\ \mathbf{F}\ X = 1 + \mathbb{N}_0 \times X \end{array}\right.$$

Abstração (listas)

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & 1 + \mathbb{N}_0 \times L \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id + id \times k"] \\
B & 1 + \mathbb{N}_0 \times B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & \mathsf{F}\ L \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "\mathsf{F}\ k"] \\
B & \mathsf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$\left\{\begin{array}{l} \mathsf{F}\ k = id + id \times k \\ \mathsf{F}\ X = 1 + \mathbb{N}_0 \times X \end{array}\right.$$

Abstração

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "\mathbf{F}\ k"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Abstração

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "\mathbf{F}\ k"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$k \cdot \mathbf{in} = g \cdot \mathbf{F}\ k$$

Abstração

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "\mathbf{F}\ k"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$k = (\!|\, g \,|\!)$$

$$k \cdot \mathsf{in} = g \cdot \mathbf{F}\ k$$

Abstração

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "\mathbf{F}\ k"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

$$k = \llparenthesis\, g\, \rrparenthesis$$

$$k \cdot \mathbf{in} = g \cdot \mathbf{F}\ k$$

$k = \llparenthesis\, g\, \rrparenthesis$ lê-se: k é o *catamorfismo* de g.

Catamorfismo

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "\llparenthesis g \rrparenthesis"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "\mathbf{F}\ \llparenthesis g \rrparenthesis"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Catamorfismo

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "(\!|\, g \,|\!)"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "\mathbf{F}\ (\!|\, g \,|\!)"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Propriedade universal:

$$k = (\!|\, g \,|\!) \quad \Leftrightarrow \quad k \cdot \mathbf{in} = g \cdot \mathbf{F}\ k$$

Sumário

Naturais: $\left\{\begin{array}{l} T = \mathbb{N}_0 \\ \mathbf{in} = [\underline{0}, succ] \\ \left\{\begin{array}{l} \mathsf{F}\ X = 1 + X \\ \mathsf{F}\ f = id + f \end{array}\right. \end{array}\right.$

Sumário

$$\text{Naturais:} \left\{ \begin{array}{l} T = \mathbb{N}_0 \\ \mathbf{in} = [\underline{0}, succ] \\ \left\{ \begin{array}{l} \mathbf{F}\; X = 1 + X \\ \mathbf{F}\; f = id + f \end{array} \right. \end{array} \right.$$

$$\mathbf{for}\; b\; i = (\!| [\underline{i}, b] |\!)$$

Sumário

$$\text{Naturais:} \begin{cases} T = \mathbb{N}_0 \\ \mathbf{in} = [\underline{0}, succ] \\ \begin{cases} \mathbf{F}\; X = 1 + X \\ \mathbf{F}\; f = id + f \end{cases} \end{cases}$$

$$\mathbf{for}\; b\; i = (\!|\, [\underline{i}, b] \,|\!)$$

$$\text{Listas:} \begin{cases} T = L \\ \mathbf{in} = [nil, cons] \\ \begin{cases} \mathbf{F}\; X = 1 + \mathbb{N}_0 \times X \\ \mathbf{F}\; f = id + id \times f \end{cases} \end{cases}$$

Sumário

Naturais: $\begin{cases} T = \mathbb{N}_0 \\ \mathbf{in} = [\underline{0}, succ] \\ \begin{cases} \mathbf{F}\ X = 1 + X \\ \mathbf{F}\ f = id + f \end{cases} \end{cases}$

$\mathbf{for}\ b\ i = (\![[\underline{i}, b]]\!)$

Listas: $\begin{cases} T = L \\ \mathbf{in} = [nil, cons] \\ \begin{cases} \mathbf{F}\ X = 1 + \mathbb{N}_0 \times X \\ \mathbf{F}\ f = id + id \times f \end{cases} \end{cases}$

$\mathbf{foldr}\ f\ e = (\![[\underline{e}, \widehat{f}]]\!)$

Exemplos já vistos

Naturais:

$$(a\times) = \llparenthesis\, [\underline{0}, (a+)]\, \rrparenthesis$$

$$a^b = \llparenthesis\, [\underline{1}, (a\times)]\, \rrparenthesis\ b$$

Listas:

$$\mathit{sum} = \llparenthesis\, [\underline{0}, \mathit{add}]\, \rrparenthesis$$

$$\mathit{length} = \llparenthesis\, [\underline{0}, \mathit{succ} \cdot \pi_2]\, \rrparenthesis$$

Cancelamento-cata

Em

$$k = (\![\, g \,]\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot \mathsf{F}\ k$$

faça-se $k := (\![\, g \,]\!)$:

Cancelamento-cata

Em

$$k = (\!| g |\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot \mathbf{F}\ k$$

faça-se $k := (\!| g |\!)$:

$$(\!| g |\!) = (\!| g |\!) \;\Leftrightarrow\; (\!| g |\!) \cdot \mathbf{in} = g \cdot \mathbf{F}\ (\!| g |\!)$$

Cancelamento-cata

Em

$$k = (\!| g |\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot \mathbf{F}\ k$$

faça-se $k := (\!| g |\!)$:

$$
\begin{array}{ll}
 & (\!| g |\!) = (\!| g |\!) \;\Leftrightarrow\; (\!| g |\!) \cdot \mathbf{in} = g \cdot \mathbf{F}\ (\!| g |\!) \\
\Leftrightarrow & \qquad \{\ x = x \Leftrightarrow \mathit{true}\ \} \\
 & \mathit{true} \;\Leftrightarrow\; (\!| g |\!) \cdot \mathbf{in} = g \cdot \mathbf{F}\ (\!| g |\!)
\end{array}
$$

Cancelamento-cata

Em

$$k = (\!| g |\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot \mathbf{F}\ k$$

faça-se $k := (\!| g |\!)$:

$$
\begin{array}{ll}
 & (\!| g |\!) = (\!| g |\!) \;\Leftrightarrow\; (\!| g |\!) \cdot \mathbf{in} = g \cdot \mathbf{F}\ (\!| g |\!) \\
\Leftrightarrow & \qquad \{\ x = x \Leftrightarrow \mathit{true}\ \} \\
 & \mathit{true} \;\Leftrightarrow\; (\!| g |\!) \cdot \mathbf{in} = g \cdot \mathbf{F}\ (\!| g |\!) \\
\Leftrightarrow & \qquad \{\ \text{trivial}\ \} \\
 & (\!| g |\!) \cdot \mathbf{in} = g \cdot \mathbf{F}\ (\!| g |\!)
\end{array}
$$

Cancelamento-cata

$$\llparenthesis\, g \,\rrparenthesis \cdot \mathbf{in} = g \cdot \mathsf{F}\, \llparenthesis\, g \,\rrparenthesis$$

$$\Leftrightarrow \qquad \{\ \text{isomorfismo } \mathbf{in} \,/\, \mathbf{out}\ \}$$

$$\llparenthesis\, g \,\rrparenthesis = g \cdot \mathsf{F}\, \llparenthesis\, g \,\rrparenthesis \cdot \mathbf{out}$$

Cancelamento-cata

$$(\!| g |\!) \cdot \mathsf{in} = g \cdot \mathsf{F}\,(\!| g |\!)$$

$$\Leftrightarrow \qquad \{ \text{ isomorfismo } \mathsf{in} \,/\, \mathsf{out} \}$$

$$(\!| g |\!) = g \cdot \mathsf{F}\,(\!| g |\!) \cdot \mathsf{out}$$

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "(\!| g |\!)"'] & \mathsf{F}\,T \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "\mathsf{F}\,(\!| g |\!)"] \\
B & \mathsf{F}\,B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Cancelamento-cata

$$
\begin{array}{ll}
& \llparenthesis\, g \,\rrparenthesis \cdot \mathsf{in} = g \cdot \mathsf{F}\ \llparenthesis\, g \,\rrparenthesis \\
\Leftrightarrow & \qquad \{\ \text{isomorfismo } \mathsf{in} \,/\, \mathsf{out}\ \} \\
& \llparenthesis\, g \,\rrparenthesis = g \cdot \mathsf{F}\ \llparenthesis\, g \,\rrparenthesis \cdot \mathsf{out}
\end{array}
$$

$$\begin{tikzcd}
T \arrow[rr, bend left, "\mathsf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "\llparenthesis\, g \,\rrparenthesis"'] & & \mathsf{F}\ T \arrow[ll, bend left, "\mathsf{in}"] \arrow[d, "\mathsf{F}\ \llparenthesis\, g \,\rrparenthesis"] \\
B & & \mathsf{F}\ B \arrow[ll, bend left, "g"]
\end{tikzcd}$$

$$\llparenthesis\, g \,\rrparenthesis = g \cdot \mathsf{F}\ \llparenthesis\, g \,\rrparenthesis \cdot \mathsf{out}$$

definição executável

Functores

O que “é exactamente“ **F**?

$$\begin{tikzcd} A \arrow[r, dotted, no head] \arrow[d, "f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\ B \arrow[r, dotted, no head] & \mathbf{F}\ B \end{tikzcd}$$

Functores

O que “é exactamente“ **F**?

$$\begin{tikzcd} A \arrow[r, dotted, no head] \arrow[d, "f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\ B \arrow[r, dotted, no head] & \mathbf{F}\ B \end{tikzcd}$$

Recordar **propriedades naturais** (grátis)

Functores

O que “é exactamente“ **F**?

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\
B \arrow[r, dotted, no head] & \mathbf{F}\ B
\end{tikzcd}$$

Por exemplo:

$$\mathbf{F}\ X = 1 + \mathbb{N}_0 \times X$$

Recordar **propriedades naturais** (grátis)

Functores

O que “é exactamente“ **F**?

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\
B \arrow[r, dotted, no head] & \mathbf{F}\ B
\end{tikzcd}$$

Recordar **propriedades naturais** (grátis)

Por exemplo:

$$\mathbf{F}\ X = 1 + \mathbb{N}_0 \times X$$

Substituir maiúsculas (X) por minúsculas ($X \to f$):

Functores

O que “é exactamente“ **F**?

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\
B \arrow[r, dotted, no head] & \mathbf{F}\ B
\end{tikzcd}$$

Recordar **propriedades naturais** (grátis)

Por exemplo:

$$\mathbf{F}\ X = 1 + \mathbb{N}_0 \times X$$

Substituir maiúsculas (X) por minúsculas ($X \rightarrow f$):

$$\mathbf{F}\ f = id + id \times f$$

Propriedades

$$\begin{aligned} \mathbf{F}\ id &= id \\ \mathbf{F}\ (g \cdot f) &= (\mathbf{F}\ g) \cdot (\mathbf{F}\ f) \end{aligned}$$

$\mathbf{F}\ id = id$

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\
B \arrow[r, dotted, no head] & \mathbf{F}\ B
\end{tikzcd}$$

$$\mathbf{F}\ id = id$$

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "id"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ id"] \\
A \arrow[r, dotted, no head] & \mathbf{F}\ A
\end{tikzcd}$$

$$\mathbf{F}\ id = id$$

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "id"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ id"] \arrow[d, bend left=60, "id"] \\
A \arrow[r, dotted, no head] & \mathbf{F}\ A
\end{tikzcd}$$

$$\mathbf{F}\ (g \cdot f) = (\mathbf{F}\ g) \cdot (\mathbf{F}\ f)$$

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\
B \arrow[r, dotted, no head] \arrow[d, "g"'] & \mathbf{F}\ B \arrow[d, "\mathbf{F}\ g"] \\
C \arrow[r, dotted, no head] & \mathbf{F}\ C
\end{tikzcd}$$

$$\mathbf{F}\ (g \cdot f) = (\mathbf{F}\ g) \cdot (\mathbf{F}\ f)$$

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "f"'] \arrow[dd, bend right=60, "g \cdot f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \\
B \arrow[r, dotted, no head] \arrow[d, "g"'] & \mathbf{F}\ B \arrow[d, "\mathbf{F}\ g"] \\
C \arrow[r, dotted, no head] & \mathbf{F}\ C
\end{tikzcd}$$

$$\mathbf{F}\ (g \cdot f) = (\mathbf{F}\ g) \cdot (\mathbf{F}\ f)$$

$$\begin{tikzcd}
A \arrow[r, dash, dotted] \arrow[d, "f"'] \arrow[dd, bend right=60, "g\cdot f"'] & \mathbf{F}\ A \arrow[d, "\mathbf{F}\ f"] \arrow[dd, bend left=60, "\mathbf{F}\ g\cdot\mathbf{F}\ f"] \arrow[dd, bend left=90, "\mathbf{F}\ (g\cdot f)"] \\
B \arrow[r, dash, dotted] \arrow[d, "g"'] & \mathbf{F}\ B \arrow[d, "\mathbf{F}\ g"] \\
C \arrow[r, dash, dotted] & \mathbf{F}\ C
\end{tikzcd}$$

Exemplo bem conhecido

Functor lista

$$\mathbf{F}\ f = f^{\star}$$

Exemplo bem conhecido

Functor lista

$$\mathbf{F}\ f = f^{\star} \quad \textbf{where} \quad f^{\star}\ x = [f\ a \mid a \leftarrow x] = \mathsf{map}\ f\ x$$

$$\begin{tikzcd}
A \arrow[r, dotted, no head] \arrow[d, "f"'] & A^{\star} \arrow[d, "f^{\star}"] \\
B \arrow[r, dotted, no head] & B^{\star}
\end{tikzcd}$$

De facto

$$id^{\star}\ x = [a \mid a \leftarrow x] = x$$

De facto

$$id^{\star}\ x = [a \mid a \leftarrow x] = x$$

$$(g^{\star} \cdot f^{\star})\ x$$

De facto

$$id^{\star}\ x = [a \mid a \leftarrow x] = x$$

$$(g^{\star} \cdot f^{\star})\ x = [g\ b \mid b \leftarrow [f\ a \mid a \leftarrow x]]$$

De facto

$$id^{\star}\ x = [a \mid a \leftarrow x] = x$$

$$(g^{\star} \cdot f^{\star})\ x = [g\ b \mid b \leftarrow [f\ a \mid a \leftarrow x]] = (g \cdot f)^{\star}\ x$$

Cálculo de Programas

Aula T07

Propriedades

Recordar

$$\begin{aligned} \mathbf{F}\ id &= id \\ \mathbf{F}\ (g \cdot f) &= (\mathbf{F}\ g) \cdot (\mathbf{F}\ f) \end{aligned}$$

Estas propriedades são essenciais para raciocinar sobre **catamorfismos**.

Ver exemplo a seguir.

Propriedade fusão-cata

$$\begin{tikzcd}
T \arrow[d, swap, "\llparenthesis g \rrparenthesis"] \arrow[dd, bend right=50, swap, "\llparenthesis h \rrparenthesis"] & & \mathbf{F}\ T \arrow[ll, swap, "\mathsf{in}"] \arrow[d, "\mathbf{F}\ \llparenthesis g \rrparenthesis"] \\
A \arrow[d, swap, "f"] & & \mathbf{F}\ A \arrow[ll, "g"] \arrow[d, "\mathbf{F}\ f"] \\
B & & \mathbf{F}\ B \arrow[ll, "h"]
\end{tikzcd}$$

Fusão-cata

$$f \cdot (\!| g |\!) = (\!| h |\!)$$

Fusão-cata

$$f \cdot (\!| g |\!) = (\!| h |\!)$$

$$\Leftrightarrow \quad \{ \text{ universal-cata para } k = f \cdot (\!| g |\!) \; \}$$

$$f \cdot (\!| g |\!) \cdot \mathbf{in} = h \cdot \mathsf{F}\, (f \cdot (\!| g |\!))$$

Fusão-cata

$$
\begin{array}{ll}
 & f \cdot \llparenthesis g \rrparenthesis = \llparenthesis h \rrparenthesis \\
\Leftrightarrow & \qquad \{ \ \text{universal-cata para } k = f \cdot \llparenthesis g \rrparenthesis \ \} \\
 & f \cdot \llparenthesis g \rrparenthesis \cdot \mathbf{in} = h \cdot \mathsf{F}\ (f \cdot \llparenthesis g \rrparenthesis) \\
\Leftrightarrow & \qquad \{ \ \text{cancelamento-cata} \ \} \\
 & f \cdot g \cdot \mathsf{F}\ \llparenthesis g \rrparenthesis = h \cdot \mathsf{F}\ (f \cdot \llparenthesis g \rrparenthesis)
\end{array}
$$

Fusão-cata

$$f \cdot g \cdot \mathsf{F}\ (\!|\, g \,|\!) = h \cdot \mathsf{F}\ (f \cdot (\!|\, g \,|\!))$$

Fusão-cata

$$
\begin{aligned}
& f \cdot g \cdot \mathbf{F}\, (\!| g |\!) = h \cdot \mathbf{F}\, (f \cdot (\!| g |\!)) \\
\Leftrightarrow \quad & \quad \{ \text{ functor-}\mathbf{F}\ (1) \ \} \\
& f \cdot g \cdot \mathbf{F}\, (\!| g |\!) = h \cdot \mathbf{F}\, f \cdot \mathbf{F}\, (\!| g |\!)
\end{aligned}
$$

Fusão-cata

$$f \cdot g \cdot \mathsf{F}\,(\!|\, g \,|\!) = h \cdot \mathsf{F}\,(f \cdot (\!|\, g \,|\!))$$

$\Leftrightarrow$ $\{$ functor-F (1) $\}$

$$f \cdot g \cdot \mathsf{F}\,(\!|\, g \,|\!) = h \cdot \mathsf{F}\, f \cdot \mathsf{F}\,(\!|\, g \,|\!)$$

$\Leftarrow$ $\{$ Leibniz $\}$

$$f \cdot g = h \cdot (\mathsf{F}\, f)$$

Fusão-cata

$$
\begin{array}{cl}
 & f \cdot g \cdot \mathbf{F}\,(\!|\, g \,|\!) = h \cdot \mathbf{F}\,(f \cdot (\!|\, g \,|\!)) \\
\Leftrightarrow & \qquad \{ \text{ functor-}\mathbf{F}\ (1) \ \} \\
 & f \cdot g \cdot \mathbf{F}\,(\!|\, g \,|\!) = h \cdot \mathbf{F}\, f \cdot \mathbf{F}\,(\!|\, g \,|\!) \\
\Leftarrow & \qquad \{ \text{ Leibniz } \} \\
 & f \cdot g = h \cdot (\mathbf{F}\, f)
\end{array}
$$

Em suma:

$$
f \cdot (\!|\, g \,|\!) = (\!|\, h \,|\!) \quad \Leftarrow \quad f \cdot g = h \cdot (\mathbf{F}\, f)
$$

O padrão $\langle f , g \rangle = (\!| h |\!)$

Catas que produzem pares:

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "(\!| g |\!)"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "\mathbf{F}\ (\!| g |\!)"] \\
A \times B & \mathbf{F}\ (A \times B) \arrow[l, bend left, "g"]
\end{tikzcd}$$

Catas que produzem pares

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "\langle f , h \rangle = (\!| g |\!)"'] & \mathbf{F}\; T \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "\mathbf{F}\; \langle f , h \rangle"] \\
A \times B & \mathbf{F}\; (A \times B) \arrow[l, bend left, "g"]
\end{tikzcd}$$

Recursividade mútua

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "\langle f,h\rangle = \llparenthesis \langle k,m\rangle \rrparenthesis"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "\mathbf{F}\ \langle f,h\rangle"] \\
A \times B & \mathbf{F}\ (A \times B) \arrow[l, bend left, "g=\langle k,m\rangle"]
\end{tikzcd}$$

Recursividade mútua

$$\begin{tikzcd}
T \arrow[rr, bend left, "\mathbf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "\langle f,h\rangle = \llparenthesis \langle k,m\rangle \rrparenthesis"'] & & \mathbf{F}\ T \arrow[ll, bend left, "\mathbf{in}"] \arrow[d, "\mathbf{F}\ \langle f,h\rangle"] \\
A \times B & & \mathbf{F}\ (A \times B) \arrow[ll, bend left, "g = \langle k,m\rangle"]
\end{tikzcd}$$

$$\langle f, h\rangle = \llparenthesis \langle k, m\rangle \rrparenthesis \Leftrightarrow \left\{ \begin{array}{l} f \cdot \mathbf{in} = k \cdot \mathbf{F}\ \langle f, h\rangle \\ h \cdot \mathbf{in} = m \cdot \mathbf{F}\ \langle f, h\rangle \end{array} \right.$$

Recursividade mútua

$$
\begin{array}{ll}
& \langle f, h \rangle = (\!| \langle k, m \rangle |\!) \\
\Leftrightarrow & \quad \{ \text{ universal-cata } \} \\
& \langle f, h \rangle \cdot \mathbf{in} = \langle k, m \rangle \cdot \mathsf{F}\ \langle f, h \rangle \\
\Leftrightarrow & \quad \{ \text{ fusão-}\times \} \\
& \langle f \cdot \mathbf{in}, h \cdot \mathbf{in} \rangle = \langle h \cdot \mathsf{F}\ \langle f, h \rangle, m \cdot \mathsf{F}\ \langle f, h \rangle \rangle \\
\Leftrightarrow & \quad \{ \text{ Eq-}\times \} \\
& \left\{ \begin{array}{l} f \cdot \mathbf{in} = k \cdot \mathsf{F}\ \langle f, h \rangle \\ h \cdot \mathbf{in} = m \cdot \mathsf{F}\ \langle f, h \rangle \end{array} \right.
\end{array}
$$

Recursividade mútua — caso $T = \mathbb{N}_0$

$$\langle f, g \rangle = (\!| \langle h, k \rangle |\!)$$

$$\Leftrightarrow \quad \{ \text{ recursividade mútua com } \mathbf{in} = [\underline{0}, succ] \text{ etc } \}$$

$$\begin{cases} f \cdot [\underline{0}, succ] = h \cdot (id + \langle f, g \rangle) \\ g \cdot [\underline{0}, succ] = k \cdot (id + \langle f, g \rangle) \end{cases}$$

$$\Leftrightarrow \quad \{ \text{ sejam } h = [h_1, h_2] \text{ e } k = [k_1, k_2] \ \}$$

$$\begin{cases} f \cdot [\underline{0}, succ] = [h_1, h_2] \cdot (id + \langle f, g \rangle) \\ g \cdot [\underline{0}, succ] = [k_1, k_2] \cdot (id + \langle f, g \rangle) \end{cases}$$

Recursividade mútua — caso $T = \mathbb{N}_0$

Continuando:

$$
\begin{array}{ll}
& \langle f , g \rangle = (\!| \langle [h_1 , h_2], [k_1 , k_2] \rangle |\!) \\
\Leftrightarrow & \qquad \{ \text{ leis dos coprodutos } \} \\
& \left\{ \begin{array}{l} \left\{ \begin{array}{l} f \cdot \underline{0} = h_1 \\ f \cdot succ = h_2 \cdot \langle f , g \rangle \end{array} \right. \\ \left\{ \begin{array}{l} g \cdot \underline{0} = k_1 \\ g \cdot succ = k_2 \cdot \langle f , g \rangle \end{array} \right. \end{array} \right.
\end{array}
$$

Recursividade mútua — caso $T = \mathbb{N}_0$

Finalmente, adicionar variáveis no lado direito, com $h_1 = \underline{a}$ e $k_1 = \underline{b}$:

$$\langle f, g \rangle = (\!| \langle [\underline{a}, h_2], [\underline{b}, k_2] \rangle |\!) \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \left\{ \begin{array}{l} f\ 0 = a \\ f\ (n+1) = h_2\ (f\ n, g\ n) \end{array} \right. \\ \left\{ \begin{array}{l} g\ 0 = b \\ g\ (n+1) = k_2\ (f\ n, g\ n) \end{array} \right. \end{array} \right.$$

Exemplo — Fibonacci

$$\begin{aligned}
&\mathit{fib}\ 0 = 1\\
&\mathit{fib}\ 1 = 1\\
&\mathit{fib}\ (n+2) = \mathit{fib}\ (n+1) + \mathit{fib}\ n
\end{aligned}$$

Claramente, não é um catamorfismo de $\mathbb{N}_0$.

Mas pode ser transformado nesse sentido.

Exemplo — Fibonacci

Vamos definir uma função auxiliar:

$$f\ n = \mathit{fib}\ (n + 1)$$

De imediato:

$$\begin{aligned} &f\ 0 = \mathit{fib}\ 1 = 1 \\ &f\ (n + 1) = \mathit{fib}\ (n + 2) = f\ n + \mathit{fib}\ n \end{aligned}$$

Exemplo — Fibonacci

Quanto a *fib* propriamente dita:

$$\begin{aligned} & \mathit{fib}\ 0 = 1 \\ & \mathit{fib}\ (n+1) = f\ \ n \end{aligned}$$

Em suma, temos a recursividade mútua:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} f\ \ 0 = \mathit{fib}\ 1 = 1 \\ f\ \ (n+1) = f\ \ n + \mathit{fib}\ n \end{array} \right. \\ \left\{ \begin{array}{l} \mathit{fib}\ 0 = 1 \\ \mathit{fib}\ (n+1) = f\ \ n \end{array} \right. \end{array} \right.$$

Exemplo — Fibonacci

Aplicando a lei a este caso:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} f\ 0 = a \\ f\ (n+1) = h_2\ (f\ \ n, \mathit{fib}\ n) \end{array} \right. \\ \left\{ \begin{array}{l} \mathit{fib}\ 0 = b \\ \mathit{fib}\ (n+1) = k_2\ (f\ \ n, \mathit{fib}\ n) \end{array} \right. \end{array} \right. \quad \Leftrightarrow \quad \langle f, \mathit{fib} \rangle = (\!|\, \langle [\underline{a}, h_2], [\underline{b}, k_2] \rangle \,|\!)$$

Exemplo — Fibonacci

Soluções:

$$\begin{aligned} a &= 1 \\ b &= 1 \\ h_2 &= add \\ k_2 &= \pi_1 \end{aligned}$$

Logo:

$$\langle f, fib \rangle = (\!|\, \langle [\underline{1}, add], [\underline{1}, \pi_1] \rangle \,|\!)$$

Exemplo — Fibonacci

Sendo $\langle f, fib \rangle$ um **catamorfismo** de $\mathbb{N}_0$, podemos convertê-lo num **ciclo-for** de acordo com a definição deste:

$$\textbf{for}\ b\ i = (\!|\, [\underline{i}, b] \,|\!)$$

Exemplo — Fibonacci

Sendo $\langle f, fib \rangle$ um **catamorfismo** de $\mathbb{N}_0$, podemos convertê-lo num **ciclo-for** de acordo com a definição deste:

$$\textbf{for}\ b\ i = (\!|\, [\underline{i}, b] \,|\!)$$

$$\begin{array}{cl}
 & \langle f, fib \rangle = (\!|\, \langle [\underline{1}, add], [\underline{1}, \pi_1] \rangle \,|\!) \\
\Leftrightarrow & \{ \text{ lei da troca } \} \\
 & \langle f, fib \rangle = (\!|\, [\langle \underline{1}, \underline{1} \rangle, \langle add, \pi_1 \rangle] \,|\!) \\
\Leftrightarrow & \{ \text{ recordar ficha 3 } \} \\
 & \langle f, fib \rangle = (\!|\, [\underline{(1, 1)}, \langle add, \pi_1 \rangle] \,|\!) \\
\Leftrightarrow & \{ \text{ definição de } \textbf{for}\ b\ i \ \} \\
 & \langle f, fib \rangle = \textbf{for}\ \langle add, \pi_1 \rangle\ (1, 1)
\end{array}$$

Exemplo — Fibonacci

In summary:

$$\mathit{fib} = \pi_2 \cdot (\mathbf{for}\ \langle \mathit{add}, \pi_1 \rangle\ (1, 1))$$

Cf. the same in C:

```
int fib(int n)
{
int x=1; int y=1; int i;
for (i=1;i<=n;i++) {int a=x; x=x+y; y=a;}
return y;
};
```

Exemplo — factorial

$$\begin{array}{l} fac\ 0 = 1 \\ fac\ (n+1) = \underbrace{n+1}_{f\ n} \times fac\ n \end{array}$$

Cf.

$$\langle f, fac \rangle = (\!|\, \langle [\underline{a}, h_2], [\underline{b}, k_2] \rangle \,|\!) \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \left\{ \begin{array}{l} f\ 0 = a \\ f\ (n+1) = h_2\ (f\ n, fac\ n) \end{array} \right. \\ \left\{ \begin{array}{l} fac\ 0 = b \\ fac\ (n+1) = k_2\ (f\ n, fac\ n) \end{array} \right. \end{array} \right.$$

Exemplo — factorial

$$\langle f, fac\rangle = (\!|\, \langle [\underline{a}, h_2], [\underline{b}, k_2] \rangle \,|\!) \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \left\{ \begin{array}{l} f\ 0 = a \\ f\ (n+1) = h_2\ (f\ n, fac\ n) \end{array} \right. \\ \left\{ \begin{array}{l} fac\ 0 = b \\ fac\ (n+1) = k_2\ (f\ n, fac\ n) \end{array} \right. \end{array} \right.$$

$k_2 = mul$

$b = 1$

$f\ n = n + 1$

$a = 1$

(Falta h_2...)

Exemplo — Fibonacci

$$\begin{aligned} & f\ (n+1) \\ = \ & (n+1)+1 \\ = \ & 1+(n+1) \\ = \ & 1+f\ \ n \\ = \ & 1+\pi_1\ (f\ \ n, fac\ \ n) \\ = \ & (succ \cdot \pi_1)\ (f\ \ n, fac\ \ n) \end{aligned}$$

Logo:

$$k_2 = mul$$

$$b = 1$$

$$h_2 = succ \cdot \pi_1$$

$$a = 1$$

Exemplo — Fibonacci

Finalmente:

$$
\begin{aligned}
& \langle f, \mathit{fac} \rangle \\
= \quad & \llparenthesis \, \langle [\underline{1}, \mathit{succ} \cdot \pi_1], [\underline{1}, \mathit{mul}] \rangle \, \rrparenthesis \\
= \quad & \llparenthesis \, [\underline{(1, 1)}, \langle \mathit{succ} \cdot \pi_1, \mathit{mul} \rangle] \, \rrparenthesis \\
= \quad & \textbf{for } \langle \mathit{succ} \cdot \pi_1, \mathit{mul} \rangle \; ((1, 1)) \\
= \quad & \textbf{for } g \; (1, 1) \textbf{ where } g \; (x, y) = (x + 1, x \times y)
\end{aligned}
$$

Ciclos-for (em C)

Extraído de uma ficha:

A função $k = \textbf{for}\ f\ i$ pode ser codificada em sintaxe C escrevendo

```
int k(int n) {
  int r=i;
  int j;
  for (j=1;j<n+1;j++) {r=f(r);}
  return r;
};
```

Ciclos-for (em C)

Tendo-se

$$fac = \pi_2\ (\textbf{for}\ g\ (1, 1)\ \textbf{where}\ g\ (x, y) = (x + 1, x \times y))$$

ter-se-á:

```
int fac(int n) {
  int x=1; int y=1;
  int j;
  for (j=1;j<n+1;j++) {y=x*y;x=x+1;}
  return y;
};
```

Cálculo de Programas

Aula T08

Banana-split

Banana-split

$$(\!| \langle \_, \_ \rangle |\!)$$

Banana-split

$$(\!|\, \langle \_ , \_ \rangle \,|\!)$$

$$\langle (\!|\, \_ \,|\!), (\!|\, \_ \,|\!) \rangle$$

Banana-split

$$\llparenthesis \langle \_, \_ \rangle \rrparenthesis$$

$$\langle \llparenthesis \_ \rrparenthesis, \llparenthesis \_ \rrparenthesis \rangle$$

$$\langle \llparenthesis i \rrparenthesis, \llparenthesis j \rrparenthesis \rangle$$

Banana-split

$$\langle (\!| i |\!), (\!| j |\!) \rangle$$

$$\begin{tikzcd}
 & & & & \mathbf{F}\,T \arrow[dlll, "\mathbf{in}"'] \arrow[ddl, "\mathbf{F}(\!| i |\!)"'] \arrow[ddd, dotted, "f_6"'] \arrow[ddddr, "\mathbf{F}(\!| j |\!)"] & \\
 & T \arrow[ddl, "(\!| i |\!)"'] \arrow[ddd, dotted, "f_7"'] \arrow[ddddr, "(\!| j |\!)"] & & & & \\
 & & & \mathbf{F}A \arrow[dlll, "i"'] & & \\
A & & & & \mathbf{F}(A \times B) \arrow[ul, "f_1"'] \arrow[llll, dotted, "f_5"'] \arrow[dlll, dotted, "f_4"'] \arrow[ddll, dotted, "f_3"] \arrow[dr, "f_2"] & \\
 & A \times B \arrow[ul, "\pi_1"] \arrow[dr, "\pi_2"'] & & & & \mathbf{F}B \arrow[dlll, "j"] \\
 & & B & & &
\end{tikzcd}$$

Banana-split

$$\langle \llparenthesis\, i \,\rrparenthesis, \llparenthesis\, j \,\rrparenthesis \rangle$$

$$\begin{tikzcd}
 & & & & \mathbf{F}\,T \arrow[dlll, "\mathbf{in}"'] \arrow[ddl, "\mathbf{F}\llparenthesis\, i \,\rrparenthesis"'] \arrow[ddd, dotted, "f_6"'] \arrow[ddddr, "\mathbf{F}\llparenthesis\, j \,\rrparenthesis"] & \\
 & T \arrow[ddl, "\llparenthesis\, i \,\rrparenthesis"'] \arrow[ddd, dotted, "f_7"'] \arrow[ddddr, "\llparenthesis\, j \,\rrparenthesis"] & & & & \\
 & & & \mathbf{F}A \arrow[dlll, "i"'] & & \\
A & & & & \mathbf{F}(A \times B) \arrow[ul, "\mathbf{F}\ \pi_1"'] \arrow[llll, dotted, "f_5"'] \arrow[dlll, dotted, "f_4"'] \arrow[ddll, dotted, "f_3"'] \arrow[dr, "\mathbf{F}\ \pi_2"'] & \\
 & A \times B \arrow[ul, "\pi_1"] \arrow[dr, "\pi_2"'] & & & & \mathbf{F}B \arrow[dlll, "j"] \\
 & & B & & &
\end{tikzcd}$$

Banana-split

$\langle \llparenthesis\, i \,\rrparenthesis, \llparenthesis\, j \,\rrparenthesis \rangle$

$$\begin{tikzcd}
 & & & & \mathbf{F}\,T \arrow[dlll, "\mathsf{in}"'] \arrow[ddl, "\mathbf{F}\llparenthesis\, i \,\rrparenthesis"'] \arrow[ddd, dotted, "f_6"'] \arrow[ddddr, "\mathbf{F}\llparenthesis\, j \,\rrparenthesis"] & \\
 & T \arrow[ddl, "\llparenthesis\, i \,\rrparenthesis"'] \arrow[ddd, dotted, "f_7"'] \arrow[ddddr, "\llparenthesis\, j \,\rrparenthesis"] & & & & \\
 & & & \mathbf{F}A \arrow[dlll, "i"'] & & \\
A & & & & \mathbf{F}(A \times B) \arrow[ul, "\mathbf{F}\ \pi_1"'] \arrow[llll, dotted, "i \cdot \mathbf{F}\ \pi_1"'] \arrow[dlll, dotted, "f_4"'] \arrow[ddll, dotted, "j \cdot \mathbf{F}\ \pi_2"'] \arrow[dr, "\mathbf{F}\ \pi_2"'] & \\
 & A \times B \arrow[ul, "\pi_1"] \arrow[dr, "\pi_2"'] & & & & \mathbf{F}B \arrow[dlll, "j"] \\
 & & B & & &
\end{tikzcd}$$

Banana-split

$\langle (\!| i |\!), (\!| j |\!) \rangle$

$$\begin{tikzcd}
 & & & & \mathbf{F}\,T \arrow[dlll, "\mathbf{in}"'] \arrow[ddl, "\mathbf{F}(\!| i |\!)"'] \arrow[ddd, dotted, "f_6"'] \arrow[ddddr, "\mathbf{F}(\!| j |\!)"] & \\
 & T \arrow[ddl, "(\!| i |\!)"'] \arrow[ddd, dotted, "f_7"'] \arrow[ddddr, "(\!| j |\!)"] & & & & \\
 & & & \mathbf{F}A \arrow[dlll, "i"'] & & \\
A & & & & \mathbf{F}(A \times B) \arrow[ul, "\mathbf{F}\ \pi_1"'] \arrow[llll, dotted, "i \cdot \mathbf{F}\ \pi_1"'] \arrow[dlll, dotted, "\langle i \cdot \mathbf{F}\ \pi_1, j \cdot \mathbf{F}\ \pi_2 \rangle"'] \arrow[ddll, dotted, "j \cdot \mathbf{F}\ \pi_2"] \arrow[dr, "\mathbf{F}\ \pi_2"] & \\
 & A \times B \arrow[ul, "\pi_1"] \arrow[dr, "\pi_2"'] & & & & \mathbf{F}B \arrow[dlll, "j"] \\
 & & B & & &
\end{tikzcd}$$

Banana-split

$\langle \llparenthesis i \rrparenthesis, \llparenthesis j \rrparenthesis \rangle$

$$\begin{tikzcd}
 & & & & & \mathbf{F}\,T \arrow[dllll, "\mathsf{in}"'] \arrow[ddll, "\mathbf{F}\llparenthesis i \rrparenthesis"'] \arrow[ddd, dotted, "\mathbf{F}\,\langle \llparenthesis i \rrparenthesis, \llparenthesis j \rrparenthesis \rangle"'] \arrow[ddddr, "\mathbf{F}\llparenthesis j \rrparenthesis"] & \\
 & T \arrow[ddl, "\llparenthesis i \rrparenthesis"'] \arrow[ddd, dotted, "\langle \llparenthesis i \rrparenthesis, \llparenthesis j \rrparenthesis \rangle"'] \arrow[ddddr, "\llparenthesis j \rrparenthesis"] & & & & & \\
 & & & \mathbf{F}A \arrow[dlll, "i"'] & & & \\
A & & & & & \mathbf{F}(A \times B) \arrow[ull, "\mathbf{F}\ \pi_1"'] \arrow[lllll, dotted, "i \cdot \mathbf{F}\ \pi_1"'] \arrow[dllll, dotted, "\langle i \cdot \mathbf{F}\ \pi_1, j \cdot \mathbf{F}\ \pi_2 \rangle"'] \arrow[ddlll, dotted, "j \cdot \mathbf{F}\ \pi_2"] \arrow[dr, "\mathbf{F}\ \pi_2"] & \\
 & A \times B \arrow[ul, "\pi_1"] \arrow[dr, "\pi_2"'] & & & & & \mathbf{F}B \arrow[dllll, "j"] \\
 & & B & & & &
\end{tikzcd}$$

Banana-split

$$\langle \llparenthesis\, i\, \rrparenthesis, \llparenthesis\, j\, \rrparenthesis \rangle$$

$$\begin{tikzcd}
 & & & & \mathbf{F}\,T \arrow[dlll, "\mathsf{in}"'] \arrow[ddl, "\mathbf{F}\llparenthesis\, i\, \rrparenthesis"'] \arrow[ddd, dotted, "\mathbf{F}\,\langle \llparenthesis\, i\, \rrparenthesis, \llparenthesis\, j\, \rrparenthesis \rangle"'] \arrow[ddddr, "\mathbf{F}\llparenthesis\, j\, \rrparenthesis"] & \\
 & T \arrow[ddl, "\llparenthesis\, i\, \rrparenthesis"'] \arrow[ddd, dotted, "\langle \llparenthesis\, i\, \rrparenthesis, \llparenthesis\, j\, \rrparenthesis \rangle"'] \arrow[ddddr, "\llparenthesis\, j\, \rrparenthesis"] & & & & \\
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 & & B & & &
\end{tikzcd}$$

Banana-split

$$\begin{tikzcd}
 & & & & \mathbf{F}T \arrow[dlll, "\mathsf{in}"'] \arrow[ddl, "\mathbf{F}(\!|i|\!)"'] \arrow[ddd, dotted, "\mathbf{F}\,\langle(\!|i|\!),(\!|j|\!)\rangle"'] \arrow[ddddr, "\mathbf{F}(\!|j|\!)"] & \\
 & T \arrow[ddl, "(\!|i|\!)"'] \arrow[ddd, dotted, "\langle(\!|i|\!),(\!|j|\!)\rangle"'] \arrow[ddddr, "(\!|j|\!)"] & & & & \\
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 & A \times B \arrow[ul, "\pi_1"] \arrow[dr, "\pi_2"'] & & & & \mathbf{F}B \arrow[dlll, "j"] \\
 & & B & & &
\end{tikzcd}$$

$$\langle (\!| i |\!), (\!| j |\!) \rangle = (\!| \langle i \cdot \mathbf{F}\ \pi_1, j \cdot \mathbf{F}\ \pi_2 \rangle |\!)$$

Exemplo

Média de uma lista (não vazia):

$$average = (/) \cdot \langle sum, length \rangle$$

onde

$$\begin{aligned} sum &= (\!|\, [0, add] \,|\!) \\ length &= (\!|\, [0, succ \cdot \pi_2] \,|\!) \end{aligned}$$

Exemplo

$$average = (/) \cdot \underbrace{\langle sum, length \rangle}_{aux}$$

Por banana-split:

$$\begin{aligned} aux &= (\!|\, \langle f_5, f_3 \rangle \,|\!) \ \textbf{where} \\ f_5 &= [0, add] \cdot (id + id \times \pi_1) \\ f_3 &= [0, succ \cdot \pi_2] \cdot (id + id \times \pi_2) \end{aligned}$$

Exemplo

Finalmente, resolvendo $aux = \llparenthesis \langle f_5, f_3 \rangle \rrparenthesis$ e convertendo tudo no fim para notação pointwise:

$$
\begin{array}{l}
average\ l = x\ /\ y\ \mathbf{where} \\
\quad (x, y) = aux\ l \\
\quad aux\ [\,] = (0, 0) \\
\quad aux\ (a : l) = (a + x, y + 1)\ \mathbf{where}\ (x, y) = aux\ l
\end{array}
$$

Exemplo

Em suma,

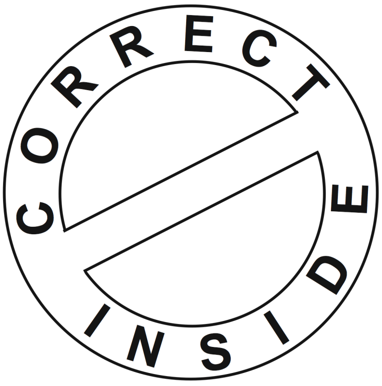
```
average l = x / y where
   (x, y) = aux l
   aux [] = (0, 0)
   aux (a : l) = (a + x, y + 1) where (x, y) = aux l
```

duas vezes mais rápida que

```
average l = (sum l) / (length l) where
   sum [] = 0
   sum (h : t) = h + sum t
   length [] = 0
   length (h : t) = 1 + length t
```

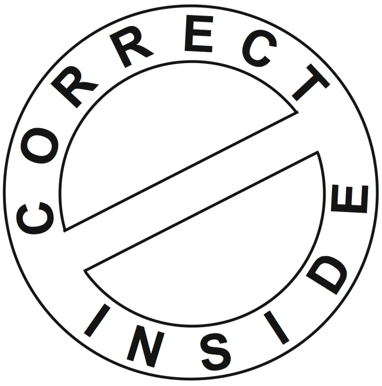
e **garantidamente** correcta.

Cálculo de Programas



Eficiência sem sacrifício da **correcção**

Cálculo de Programas



Eficiência sem sacrifício da **correcção**

A quem interessa um programa eficiente que dê resultados errados?

Recapitulação

Tipo indutivo T e o functor F que determina o seu padrão de recursividade:

$$\begin{tikzcd} T \arrow[r, bend left, "\mathsf{out}"] \arrow[r, phantom, "\cong"] & \mathsf{F}\ T \arrow[l, bend left, "\mathsf{in}"] \end{tikzcd}$$

Recapitulação

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Já vimos $T = \mathbb{N}_0$, $T = A^\star$ etc.

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Já vimos $T = \mathbb{N}_0$, $T = A^{\star}$ etc.

Vejamos mais exemplos.

Árvores binárias

Árvores com informação de tipo A nos nós, por exemplo

$$\texttt{data}\ \mathit{BTree} = \mathit{Empty} \mid \mathit{Node}\ (A, (\mathit{BTree}, \mathit{BTree}))$$

(A assumido pré-definido.)

Árvores binárias

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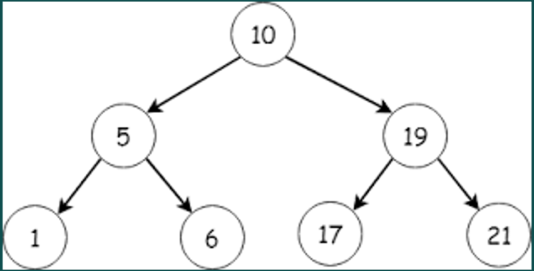
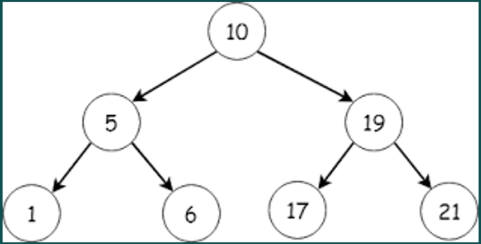
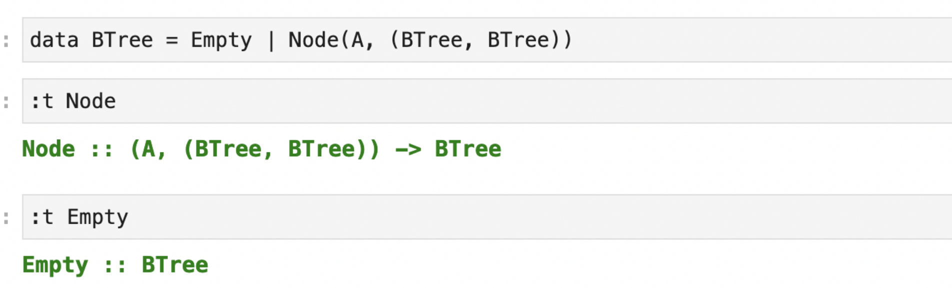


Árvores binárias



```
Node (10,
    (Node (5,
        (Node (1,
            (Empty,Empty)),
         Node (6,
            (Empty,Empty)))),
     Node (19,
         (Node (17,
            (Empty,Empty)),
          Node (21,
            (Empty,Empty))))))
```

Árvores binárias



```
data BTree = Empty | Node(A, (BTree, BTree))
```

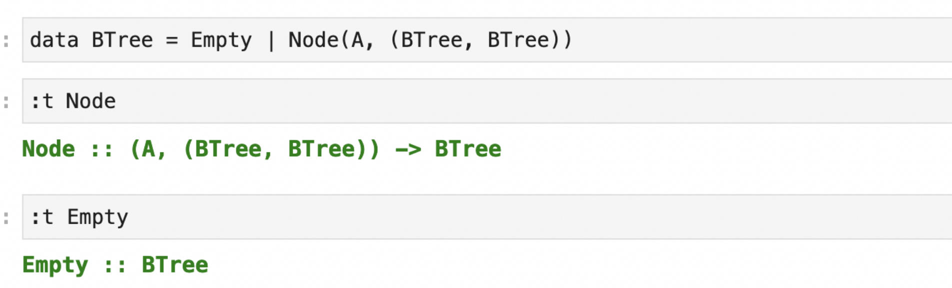
```
:t Node
```

```
Node :: (A, (BTree, BTree)) -> BTree
```

```
:t Empty
```

```
Empty :: BTree
```

Árvores binárias



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data BTree = Empty | Node(A, (BTree, BTree))
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```
:t Node
```

```
Node :: (A, (BTree, BTree)) -> BTree
```

```
:t Empty
```

```
Empty :: BTree
```

$$Node : A \times (BTree \times BTree) \rightarrow BTree$$

$$\underline{Empty} : 1 \rightarrow BTree$$

Árvores binárias

$$\mathbf{in} = [\underline{\mathit{Empty}}, \mathit{Node}]$$

Árvores binárias

$$\mathbf{in} = [\underline{Empty}, Node]$$

$$\begin{tikzcd}
BTree \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & \underbrace{1 + A \times (BTree \times BTree)}_{\mathbf{F}\ BTree} \arrow[l, bend left, "\mathbf{in}"]
\end{tikzcd}$$

$$\texttt{data}\ BTree = Empty \mid Node\ (A, (BTree, BTree))$$

Árvores binárias

$$\begin{tikzcd}
BTree \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] & \underbrace{1 + A \times (BTree \times BTree)}_{\mathbf{F}\ BTree} \arrow[l, bend left, "\mathbf{in}"]
\end{tikzcd}$$

$$T = BTree$$

$$\begin{cases} \mathbf{F}\ X = 1 + A \times (X \times X) \\ \mathbf{F}\ f = id + id \times (f \times f) \end{cases}$$

$$\mathbf{in} = [\underline{Empty}, Node]$$

Catamorfismos de árvores binárias

$$\begin{tikzcd}
\mathit{BTree} \arrow[rr, bend left, "\mathbf{out}"] \arrow[rr, phantom, "\cong"] \arrow[d, "\llparenthesis g \rrparenthesis"'] & & \underbrace{1 + A \times (\mathit{BTree} \times \mathit{BTree})}_{\mathbf{F}\ \mathit{BTree}} \arrow[ll, bend left, "\mathbf{in}"] \arrow[d, "\underbrace{id + id \times \llparenthesis g \rrparenthesis \times \llparenthesis g \rrparenthesis}_{\mathbf{F}\ \llparenthesis g \rrparenthesis}"] \\
B & & \underbrace{1 + A \times (B \times B)}_{\mathbf{F}\ B} \arrow[ll, bend left, "g"]
\end{tikzcd}$$

Catamorfismos de árvores binárias

Em Haskell:

```
cataBTree g = g . (recBTree (cataBTree g)) . outBTree

outBTree Empty              = Left ()
outBTree (Node (a,(t1,t2))) = Right(a,(t1,t2))

recBTree f = id -|- id >< (f >< f)

inBTree = either (const Empty) Node
```

Catamorfismos de árvores binárias

Exemplo: função

$$f : BTree \rightarrow \mathbb{N}_0$$

que calcula o tamanho da árvore (número total de nós).

Catamorfismos de árvores binárias

Exemplo: função

$$f : BTree \to \mathbb{N}_0$$

que calcula o tamanho da árvore (número total de nós).

Basta preocuparmo-nos com o "gene" g em

$$f = (\!| g |\!)$$

onde

$$\mathbb{N}_0 \xleftarrow{g} 1 + A \times (\mathbb{N}_0 \times \mathbb{N}_0)$$

Catamorfismos de árvores binárias

$$\begin{tikzcd}\mathbb{N}_0 & 1 + A \times (\mathbb{N}_0 \times \mathbb{N}_0) \arrow[l, "g"']\end{tikzcd}$$

Catamorfismos de árvores binárias

$$\begin{tikzcd}\mathbb{N}_0 & 1 + A \times (\mathbb{N}_0 \times \mathbb{N}_0) \arrow[l, "g = [g_1 , g_2]"']\end{tikzcd}$$

Catamorfismos de árvores binárias

árvore
vazia

$$\begin{tikzcd} \mathbb{N}_0 & 1 + A \times (\mathbb{N}_0 \times \mathbb{N}_0) \arrow[l, "g = {[g_1\, , \, g_2]}"'] \end{tikzcd}$$

$$g_1\ \_ = 0$$

Catamorfismos de árvores binárias

árvore
vazia

$$\begin{tikzcd}\mathbb{N}_0 & 1 + A \times (\mathbb{N}_0 \times \mathbb{N}_0) \arrow[l, "{g = [g_1 , g_2]}"']\end{tikzcd}$$

$$g_1 \ \_ = 0$$

tamanho

total

tamanho
sub-árvore
esquerda

tamanho
sub-árvore
direita

Catamorfismos de árvores binárias

árvore
vazia

$$\begin{tikzcd} \mathbb{N}_0 & 1 + A \times (\mathbb{N}_0 \times \mathbb{N}_0) \arrow[l, "g = [g_1\,,\,g_2]"'] \end{tikzcd}$$

$$g_1\ \_ = 0$$

$$g_2\ (a, (l, r)) = 1 + l + r$$

tamanho

total

tamanho
sub-árvore
esquerda

tamanho
sub-árvore
direita

Catamorfismos de árvores binárias

Em suma,

$$\begin{aligned} & f = (\!|\, [g_1\,, g_2] \,|\!) \ \textbf{where} \\ & \quad g_1 \ \_ = 0 \\ & \quad g_2 \ (a, (l, r)) = 1 + l + r \end{aligned}$$

Catamorfismos de árvores binárias

Em suma,

$$
\begin{aligned}
& f = (\![[g_1\,, g_2]]\!) \textbf{ where} \\
& \quad g_1 \;\_ = 0 \\
& \quad g_2\;(a, (l, r)) = 1 + l + r
\end{aligned}
$$

Haskell:

```
size = cataBTree(either g1 g2) where
   g1 _ = 0
   g2(a,(l,r)) = 1 + l + r
```

Catamorfismos em geral

Universal-cata:

$$k = (\!|\, g \,|\!) \;\Leftrightarrow\; k \cdot \mathbf{in} = g \cdot (\mathbf{F}\ k)$$

$$\begin{tikzcd}
T \arrow[r, bend left, "\mathbf{out}"] \arrow[r, phantom, "\cong"] \arrow[d, "k"'] & \mathbf{F}\ T \arrow[l, bend left, "\mathbf{in}"] \arrow[d, "\mathbf{F}\ k"] \\
B & \mathbf{F}\ B \arrow[l, bend left, "g"]
\end{tikzcd}$$

Cálculo
de
Programas

Aula T09